Computable topology and topological groups

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May 24, 2022

CCA 2022

In my talk:

1. Foundations of computable topology
2. Unexpected applications of effective algebra
3. Unexpected applications of topological groups

The central challenges:

What is the ‘right’ notion of a computable space in topology?

I will offer some answers

I have a bunch of cool fresh results supporting my claims

For simplicial complexes:

- Undecidability of the homeomorphism (Markov 1958)
- For 2-surfaces homeomorphism is decidable (well-known)
- Also decidable for 3-manifolds (essentially Perelman)
- Undecidable for a fixed *n*-manifold ($n \geq 5$) (Novikov, 1970-s).

Not every space has a triangulation.

GOAL: Develop a framework for general spaces.

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Case study: compact Polish(able) spaces.

- a computable topological space:

$$U_i \cap U_j = \bigcup_{k \in W_{f(i,j)}} U_k.$$

- Computable Polish (computably, completely metrized):

$$d(x_i, x_j) \text{ are uniformly computable for a dense } (x_i)_{i \in \omega}.$$

- Effectively compact: can additionally list all finite covers.
- Some ad hoc variations:
  1. right-c.e. effectively compact,
  2. computable topological with strong inclusions,
  3. incomplete metric,
  4. Π^0_1 in some computable Polish space, etc.

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Classics in algebra:

1. (Novikov, Boone) A f.p. group with undecidable $=$.
2. (Feiner) C.e. presented BA with no computable copy.
3. (Khisamiev) Every c.e. presented TFAG has computable copy.

Can we prove similar results in computable topology?

For that, we need to study spaces up to homeomorphism.

Here is what was known *up to homeomorphism:*

– J. Miller 2002: A Π^0_1 $C \cong_{hom} S^n$ (n-spere) in $\mathbb{R}^n$ is computable.

– Recently extended by Iljazovich et al to certain manifolds.

– Bosserhoff and Hertling 2015: Π^0_1 compact $C \subseteq \mathbb{R}^n$, s.t. $f(C)$ is not computable for any $f \in Hom(\mathbb{R}^n, \mathbb{R}^n)$

The BH2015 result is the closest to what we need.

Remark: BH2015 rediscovered "limitwise monotonic functions" central to modern effective algebra!

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A lot of progress has been done beginning with:

– A short manifesto by Selivanov (LJMath 2020)

– Independently, a paper by Harrison-T, M, Ng (JSL 2020)

– A preprint by Hoyrup, Kihara, and Selivanov (unpublished?)

We first need to separate these notions *up to homeomorphism*.

Theorem (HKS 2020; H-TMN 2020; LMN 2021; BH-TM 2021)

Effectively compact

$\downarrow \not\uparrow$

Computably metrized

$\downarrow \not\uparrow$

Computable topological

'Being a finite simplicial complex' is the strongest. (More later.)

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$$\text{Computable topological}$$

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‘Computable topological’ vs ‘computable Polish’

Theorem (BH-TM. 2021, H-TMN 2020, HKS2020)

For a Stone space S, TFAE:

- S is computably metrizable;
- S has an effectively compact copy;

but also:

- The Boolean algebra $\widehat{S}$ has a computable copy;
- $C(S; \mathbb{R})$ (is linearly isometric to) a computable Banach space.

Question (McNicholl)

For a compact Polish K, is $C(K; \mathbb{R})$ computably presentable iff K is homeomorphic to a computable Polish space?

For (separable) Stone spaces we have:

Effectively compact

$\downarrow\uparrow$

Computably metrizable

What about computable topological vs. computably metrizable?

Theorem (BH-TM 2021,Ying-Ying Tran 2018)

For a Stone space S, TFAE:

- S is homeomorphic to a Π^0_1 class;
- S is effectively compact wrt a complete right-c.e. metric;
- The Boolean algebra $\widehat{S}$ is c.e.-presentable.

Right-c.e. metrized spaces are computable topological spaces.

By the aforementioned result of Feiner, for Stone spaces:

Effectively compact

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Computably metrized

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Computable topological

Corollary (Improving BH2015 and MH-TN 2020)

There is a Π^0_1 subset of $[0, 1]$ not homeomorphic to any computable space.

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What about ‘effectively compact’ vs ‘computably metrized’?

There are several ways to build an example

All known counter-examples require effort

I will give a *connected* example using topological groups

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I will use *Pontryagin duality* $G \to \widehat{G}$ between discrete and compact abelian groups.

In the connected case:

$$\widehat{G} \cong H^1(G; \mathbb{Z}),$$

the first Čech cohomology group of the **space** of G.

A very powerful tool (M 2018, NLM 2021, and M 2021):

For a torsion-free abelian G:

$$G \text{ is computably presentable } \iff \widehat{G} \text{ is eff. compact}$$

and

$$G \text{ is } \Delta^0_2\text{-presentable } \iff \widehat{G} \text{ is computably metrizable}$$

for a p-divisible G (p is any prime).

Using an old result due to Mal'cev (1958 or 1959):

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Our proofs of the effective Pontryagin dualities use:

1. Computable bases in TFAGs (Dobrica)
2. A new constructive version of Čech cohomology
3. Priority constructions
4. Khisamiev's result about c.e. presented groups

The dualities ‘open the gates’ between computable topology and effective algebra.

A sample result:

Corollary (M 2021)

There is a connected topological space whose degree spectrum *up to homeomorphism* is (precisely) **non**-**low$_2$**.

Can do **non**-**low** for compact presentations.

Question

Can we have exactly the non-computable degrees?

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The homeomorphism problem for connected compact Polish spaces is Σ_1^1-complete.

This follows from the work of Downey and Montalban for TFAGs and the effective dualities.

And so on and so forth

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So we have:

$$\text{Effectively compact}$$
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$$\text{Computably metrized}$$
$$\downarrow \;\not\Uparrow$$
$$\text{Computable topological}$$

The dualities (and other results) seem to suggest that ‘effectively compact’ is the right notion of computability in the compact case.

What can we do with these definitions?

Theorem (H-T M 2021)

Any compact 2-surface S admits an arithmetical characterization among all Polish spaces:

$$\{i : \overline{M_i} \cong S\} \text{ is an arithmetic set.}$$

Proof: Use 22 Turing jumps (!!!) to reconstruct an atlas.

Under very mild conditions this really becomes $0''$ or so.

We also get an arithmetic triangulation.

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Computable topological groups

We really have two well-established notions:

1. Computable (constructive) discrete group (Malcev, Rabin)
2. Recursive profinite group (LaRoche, Smith):

$$F_0 \leftarrow_{\phi_0} F_1 \leftarrow_{\phi_1} F_2 \leftarrow_{\phi_2},$$

where the finite F_i and the surjective ϕ_i are given uniformly (as finite sets)

Lemma 1 (DM 2022)

For a profinite group G, TFAE:

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In the abelian case:

Theorem 2 (M 2018)

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As we have seen, this also extends to the connected case.

There are more results (Haar measure, categoricity, etc.) that we omit.

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What if our group is neither discrete nor compact?

Case study: t.d.l.c. groups

1. The domain is paths through a locally compact tree T.
2. Locally compact means T is eventually finitely branching.
3. Such groups have only countably many clopen cosets.
4. They can be realised as closed subgroups of S_∞.

Three potential definitions of computability (MN 2022):

1. Take a computably branching tree T and make operations computable on $[T]$;
2. Say that the cosets form a computable (discrete) algebraic structure;
3. Say that G is a "nicely computable subgroup" of S_∞.

Also, we could use a version of "eff. local compactness".

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Theorem (MN 2022)

All of these definitions are equivalent!!

Theorem (MN 2022)

Our definition generalizes the discrete (Rabin, Malcev) and profinite (LaRoche, Smith) cases.

Furthermore, such groups are closed under various operations such as taking the quotient, the product, etc.

The correspondence between the coset and the tree-based approach is a duality-type result.

We also have:

Theorem (LMN 2021)

Pontryagin duality is computable for abelian tdlc groups.

We can now access various decidability questions such as the decidability of the **scale function** etc.

I have no doubt our approach is the ‘right’ definition of computability for tdlc groups.

Various notions of computability can be separated for groups.

And there are classification-type questions we could study (e.g., the isomorphism problem, etc.)

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