On the computational properties of basic mathematical notions

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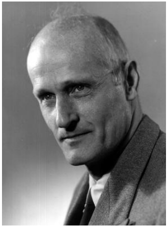
We sketch an alternative formulation of S1-S9 based on the λ-calculus and fixed point operators.

Turing and Kleene



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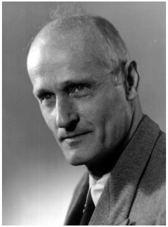

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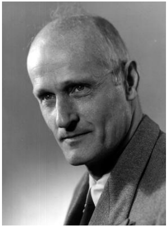

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S1-S9-computability extends Turing computability; the latter is restricted to X, Y being real numbers.

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- Equivalent λ-calculus formulation of S1-S9.
- S9 is replaced by (least) fixed point operators.
- Partial objects are allowed as oracles (essential); never as arguments.

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Historically, the focus has been on 'normal' functionals (which compute $\exists^2$ or $\exists^3$).

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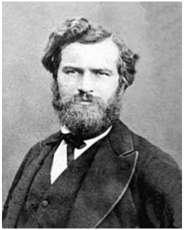
The functional $\exists^3$ computes Ω_{fin}, but no S_k^2 can compute Ω_{fin}.

Bounded variation

Jordan (1881) introduced the notion of bounded variation (BV) to generalise Dirichlet's convergence thms for Fourier series (1832).



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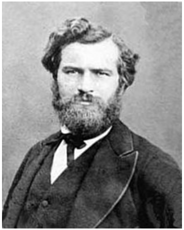

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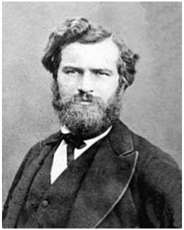

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Jordan (1881 and 1892) proves interesting/central properties of BV-functions. This includes the result that BV is exactly the class of rectifiable functions (going back to 1833-1866).

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- The function $f : [a, b] \to \mathbb{R}$ has bounded variation on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that

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(b) The total variation is then defined as:

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Jordan decomposition theorem

Jordan (1881) proves the following central theorem.

Theorem (Jordan decomposition theorem, JDT)

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A Jordan realiser is any functional that takes as input a *BV*-function f and outputs g, h as in JDT.

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Why? Because the following set is finite for BV-functions:

$$X_k := \{x \in [0,1] : |f(x+) - f(x)| > \tfrac{1}{2^k} \vee |f(x-) - f(x)| > \tfrac{1}{2^k}\}.$$

Note that BV-functions are regulated, i.e. left and right limits $f(x-)$ and $f(x+)$ exist (and are computable from $\exists^2$).

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If we additionally add a well-ordering of $[0,1]$, then we can compute $\exists^3$.

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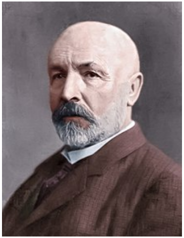
The functional Ω_1 is less explosive than Ω_{fin}: combined with the Suslin functional, Ω_1 computes Halting problem for ITTMs.

The uncountability of $\mathbb{R}$

Cantor's first set theory paper (1874): uncountability of $\mathbb{R}$.



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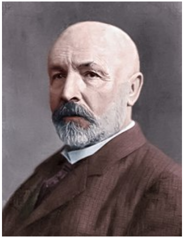

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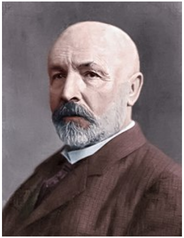

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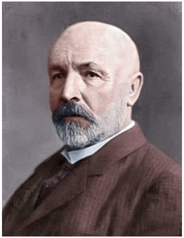

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The following definition does yield many equivalences:

Definition (see [The Literature])

A set $A \subset \mathbb{R}$ is weakly countable if there is a height $H : \mathbb{R} \to \mathbb{N}$ such that $A_n := \{x \in [0, 1] : H(x) < n\}$ is finite for all $n \in \mathbb{N}$.

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The following definition amounts to 'union over $\mathbb{N}$ of finite sets'.

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Final Thoughts

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Any (content) questions?

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for this document.