Weihrauch Reducibility on Assemblies

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Recap

- Weihrauch Reducibility:
  - Preorder on multi-functions betw. represented spaces
- Two versions:
  - ordinary Weihrauch reducibility $\leq_{\mathrm{W}}$
  - strong Weihrauch reducibility $\leq_{\mathrm{sW}}$
- $f \leq_{\mathrm{W}} g$ and $f \leq_{\mathrm{sW}} g$ formalise

  "f can be computed with the help of g."
- Two equivalent ways to define $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$:
  (1) a realizer-based definition
  (2) an abstract definition

In this talk
Assembly $=$ a set endowed with a multi-representation

Aim
Lift $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$ to multi-functions on assemblies (in two ways)

Observation
The realizer-based definition and the abstract definition differ.

Positive Result
Any new Weihrauch degree contains a single-valued problem.

Assemblies

Recap

- A *multi-function* or *problem* f is a relation between sets X and Y,
- written as $f :\subseteq X \rightrightarrows Y$.
- X is the *input space*, Y is the *output space* of f.
- Notations:
  - $f[x] := \{y \in \mathsf{Y} \,|\, (x, y) \in f\}$
  - $\mathrm{dom}(f) := \{x \in \mathsf{X} \,|\, f[x] \neq \emptyset\}$

Definition

A *multi-representation* of X is a partial multi-function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ that is *surjective*, i.e.:

$$\forall x \in X. \exists p \in \mathbb{N}^{\mathbb{N}}. x \in \delta[p]$$

Definition

An *assembly* or *multi-represented space* X is

- a set X
- endowed with a multi-representation $\delta_{\mathsf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$,
- written as pair $\mathsf{X} = (X, \delta_{\mathsf{X}})$.

Remark

- The literature considers also other name-spaces than $\mathbb{N}^{\mathbb{N}}$.
- A *multi-numbering* is a surjective multi-function $\nu :\subseteq \mathbb{N} \rightrightarrows X$.
- There is a notion of *admissibility* for multi-representations.

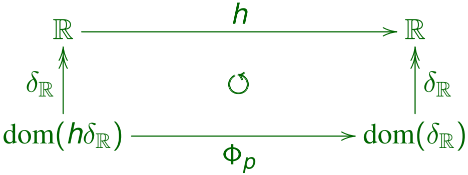
Example (Partial continuous functions)

- Let $\mathsf{P}(\mathbb{R},\mathbb{R}) := \{\text{all partial continuous functions on } \mathbb{R}\}$.
- Define a multi-representation by

$$\delta_{\mathsf{P}(\mathbb{R},\mathbb{R})}[p] \ni h \;:\Longleftrightarrow\; \text{the function } \Phi_p \text{ realizes } h$$

where Φ is the standard representation of all continuous functions $H\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ with $\mathcal{G}_\delta$-domain.

- Diagram:



- Any name for h is also a name for any restriction of h.

Example (Using multi-numberings to construct representations)

Let $\mathrm{X} = \big(X, d, (\alpha_i)_{i\in\mathbb{N}}\big)$ be a computable metric space.

- For $m \in \mathbb{N}$ define $\nu_m\colon \mathbb{N} \rightrightarrows X$ by

$$\nu_m[k] \ni x \;:\Longleftrightarrow\; d\big(\alpha_k, x\big) < 2^{-m}$$

- The meet $\bigwedge_{m\in\mathbb{N}} \nu_m :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$
  - is single-valued
  - and essentially the Cauchy representation for X.
- This yields a simple proof of admissibility of the Cauchy representation, because:

Proposition

Any multi-numbering $\nu :\subseteq \mathbb{N} \rightrightarrows Y$ is admissible (w.r.t. the convergence relation induced by ν on Y).

Example (Admissible Completion)

Let $\mathbf{Y}$ be a multi-represented space.

- Define the *completion* $\widetilde{\mathbf{Y}}$ by
  - Underlying set: $\mathbf{Y} \cup \{\uparrow\}$
  - Multi-representation: $\delta_{\widetilde{\mathbf{Y}}}[p] := \{\uparrow\} \cup \delta_{\mathbf{Y}}[p-1]$

    where $p-1 \in \mathbb{N}^* \cup \mathbb{N}^{\mathbb{N}}$ is obtained
    - by first erasing all 0's
    - and then taking -1 componentwise.
- Then $\delta_{\widetilde{\mathbf{Y}}}$ is admissible, if $\delta_{\mathbf{Y}}$ is.
- The usual completion $\overline{\mathbf{Y}}$ is not admissible in general.

Realizable Multi-functions between Assemblies

Up to now

Only defined:

- realizability of *multi*-functions w.r.t. representations
- realizability of functions w.r.t. *multi*-representations

Recap

$F\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ realizes:

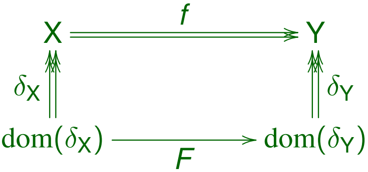
- a *multi*-function f w.r.t. representations δ_X, δ_Y, if
  - $\delta_Y(F(p)) \in f[x]$
- a function f w.r.t. *multi*-representations δ_X, δ_Y, if
  - $\delta_Y[F(p)] \ni f(x)$

for all p, x with $x \in \delta_X[p] \cap \mathrm{dom}(f)$.

Definition

Let $f :\subseteq \mathsf{X} \rightrightarrows \mathsf{Y}$ be a multi-function on multi-represented spaces.

- We call F a *realizer* for f, if
  - $\delta_{\mathsf{Y}}[F(p)] \cap f[x] \neq \emptyset$ for all p, x with $x \in \delta_{\mathsf{X}}[p] \cap \mathrm{dom}(f)$.
- Diagram (unprecise!):



- Notation: $F \vdash f$
- f is *computable*, if f has a computable realizer.

Remark

$F \vdash f$ means:

p name of $x \implies F(p)$ name of some $y \in f[x]$.

Warning

Not every function f between assemblies has a realizer:

- Counterexample:
  - Let x_1, x_2 have a common name p,
  - but $f(x_1)$, $f(x_2)$ don't.
  - Then there is no option to define $F(p)$,
  - because $f(x_1), f(x_2) \in \delta_Y[F(p)]$ is impossible.

Suggestion

Define a *problem* to be a multi-function that has a realizer.

Example (Computable Functions on Assemblies)

- The evaluation function $\mathit{eval}\colon \mathsf{P}(\mathbb{R},\mathbb{R}) \times \mathbb{R} \dashrightarrow \mathbb{R}$.
- The operator $\underbrace{h}_{\in \mathsf{P}(\mathbb{R},\mathbb{R})} \mapsto \underbrace{2 \cdot h}_{\in \mathsf{P}(\mathbb{R},\mathbb{R})}$.

Non-Example

The multi-function $\mathrm{dom} :\subseteq \mathsf{P}(\mathbb{R},\mathbb{R}) \rightrightarrows \mathbb{R}$

$$h \mapsto \mathrm{dom}(h)$$

is not computable, not even realizable.

Lifting Weihrauch Reducibility to Assemblies

Plan

We will define

- realizer-based generalisations $\leq_{\mathrm{rW}}$ and $\leq_{\mathrm{srW}}$
- abstract generalisations $\leq_{\mathrm{aW}}$ and $\leq_{\mathrm{saW}}$

of $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{sW}}$.

Idea

We just take

- the classical definition and
- its characterisation via computable multi-functions

(cf. the handbook paper by V. Brattka, G. Gherardi, A. Pauly).

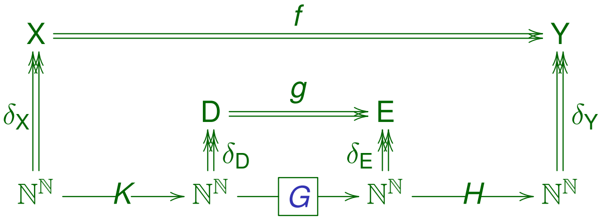
Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq D \rightrightarrows E$ be multi-functions on assemblies.

Definition

- f is *strongly r-Weihrauch-reducible* to g, if there are computable functions $K, H : \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ such that

$$G \vdash g \implies HGK \vdash f$$

- Diagram:



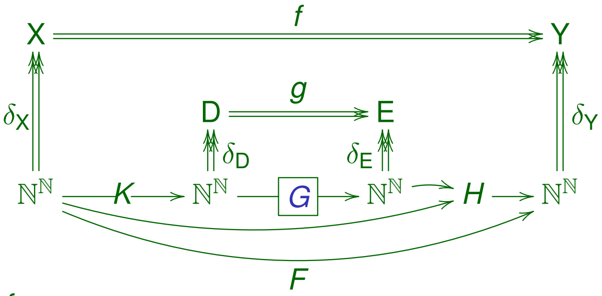
- Notation: $f \leq_{\mathrm{srW}} g$

Definition

- f is *r-Weihrauch-reducible* to g, if there are computable functions $K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $H :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that

$$G \vdash g \implies \underbrace{H(id, GK)}_{F} \vdash f$$

- Diagram:



- Notation: $f \leq_{\mathrm{rW}} g$

Definition

- f is *strongly a-Weihrauch-reducible* to g, if there are computable $k :\subseteq X \rightrightarrows D$ and $h :\subseteq E \rightrightarrows Y$ such that

$$\emptyset \neq h \circ g \circ k[x] \subseteq f[x]$$

for all $x \in \mathrm{dom}(f)$.

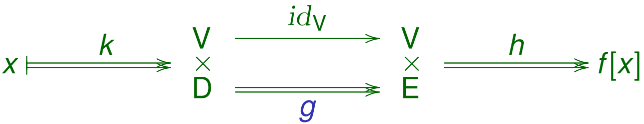
- Diagram:

$$x \overset{k}{\Longmapsto} D \overset{g}{\Longrightarrow} E \overset{h}{\Longrightarrow} f[x]$$

- Notation: $f \leq_{\mathrm{saW}} g$

Definition

- f is *a-Weihrauch-reducible* to g, if there is an assembly V and partial computable $k :\subseteq \mathsf{X} \rightrightarrows \mathsf{V} \times \mathsf{D}$, $h :\subseteq \mathsf{V} \times \mathsf{E} \rightrightarrows \mathsf{Y}$ such that
$$\emptyset \neq h \circ (id_{\mathsf{V}} \times g) \circ k[x] \subseteq f[x]$$
for all $x \in \mathrm{dom}(f)$.
- Diagram:



- Notation: $f \leq_{\mathrm{aW}} g$

Remark

As V one may take: $\mathrm{Graph}(\delta_{\mathsf{X}}) \times \mathsf{D}$

Lemma

- $\leq_{\mathrm{rW}}, \leq_{\mathrm{srW}}, \leq_{\mathrm{aW}}, \leq_{\mathrm{saW}}$ are reflexive and transitive.
- $f \leq_{\mathrm{srW}} g \implies f \leq_{\mathrm{rW}} g$
- $f \leq_{\mathrm{saW}} g \implies f \leq_{\mathrm{aW}} g$

Fact

For problems s, t on represented spaces:

- $s \leq_{\mathrm{rW}} t \iff s \leq_{\mathrm{aW}} t \iff s \leq_{\mathrm{W}} t$
- $s \leq_{\mathrm{srW}} t \iff s \leq_{\mathrm{saW}} t \iff s \leq_{\mathrm{sW}} t$

Realizer-based Definition versus Abstract Definition

Lemma

- $f \leq_{\mathrm{aW}} g \implies f \leq_{\mathrm{rW}} g$
- $f \leq_{\mathrm{saW}} g \implies f \leq_{\mathrm{srW}} g$

Proposition

Let $f :\subseteq \mathsf{X} \rightrightarrows \mathsf{Y}$ and $g :\subseteq \mathsf{D} \rightrightarrows \mathsf{E}$ be multi-functions.

- If δ_{D} is single-valued and g is realizable, then
  - $f \leq_{\mathrm{aW}} g \iff f \leq_{\mathrm{rW}} g$.
- If δ_{Y}, δ_{D} are single-valued and g is realizable, then
  - $f \leq_{\mathrm{saW}} g \iff f \leq_{\mathrm{srW}} g$.

Proposition

- $\leq_{\mathrm{aW}}$ is strictly finer than $\leq_{\mathrm{rW}}$.
- $\leq_{\mathrm{saW}}$ is strictly finer than $\leq_{\mathrm{srW}}$.

Proposition

$f \leq_{\mathrm{srW}} g \not\Longrightarrow f \leq_{\mathrm{aW}} g$,

even if

- f, g are single-valued,

& $\mathrm{dom}(f)$, $\mathrm{codom}(f)$ are endowed with numberings,

& $\mathrm{dom}(g)$, $\mathrm{codom}(g)$ are endowed with multi-numberings.

Example

- Choose an undecidable set $M \subseteq \mathbb{N}$.
- Let $f\colon \mathbb{N} \to \{0,1\}$ be the characteristic function of M.
- Let g be the identity on $\mathbb{N} \times \{0,1,2\}$.
- Equip $\mathsf{D} := \mathrm{dom}(g)$ with the multi-numbering ν_{D}:
  - $\nu_{\mathsf{D}}[3m+0] := \{(m,0),(m,1)\}$
  - $\nu_{\mathsf{D}}[3m+1] := \{(m,1),(m,2)\}$
  - $\nu_{\mathsf{D}}[3m+2] := \{(m,2),(m,0)\}$
- Equip $\mathsf{E} := \mathrm{codom}(g)$ with the multi-numbering ν_{E}:

$$\nu_{\mathsf{E}}[3m+i] := \begin{cases} \nu_{\mathsf{D}}[3m+i] & \text{if } m \in M \\ \nu_{\mathsf{D}}[3m+(i-1) \bmod 3] & \text{else} \end{cases}$$

- Thus $\nu_{\mathsf{E}}[3m+i] = \nu_{\mathsf{D}}[3m+i] \iff m \in M$ for every $i \leq 2$.

Then $f \leq_{\mathrm{srW}} g$, but $f \not\leq_{\mathrm{aW}} g$.

Proof of $f \leq_{\mathrm{srW}} g$:

- The only realizer of g is $G\colon \mathbb{N} \to \mathbb{N}$ with
$$G(3m+i) := \begin{cases} 3m+i & \text{if } m \in M \\ 3m + (i-1) \bmod 3 & \text{else.} \end{cases}$$
for $m \in \mathbb{N}$, $i \in \{0,1,2\}$.
- Then $K, H\colon \mathbb{N} \to \mathbb{N}$ defined by
  - $K(m) := 3m$,
  - $H(3m+j) := \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else.} \end{cases}$

witness $f \leq_{\mathrm{srW}} g$.

Why $f \not\leq_{\mathrm{saW}} g$?

- Assume $k\colon \mathbb{N} \to \mathbb{N} \times \{0,1,2\}$ and $h\colon \mathbb{N} \times \{0,1,2\} \rightrightarrows \{0,1\}$ were computable such that $hgk[x] = \{f(x)\}$.
  (We can assume k to be a function, as $\mathrm{dom}(f) = (\mathbb{N}, id_{\mathbb{N}})$.)
- Let $K, H\colon \mathbb{N} \to \mathbb{N}$ be their computable realizers.
- Then we could decide M as follows:
  (1) On input $x \in \mathbb{N}$, compute $m := \lfloor K(x)/3 \rfloor$.
  % Then $gk(x) = k(x) \in \{(m,0),(m,1),(m,2)\}$.
  % $\{3m, 3m+1, 3m+2\}$ contains two ν_{E}-names of $gk(x)$.
  (2) Compute in parallel $H(3m), H(3m+1), H(3m+2)$, until two computations terminate with the same $b \in \{0,1\}$.
  (3) Output b.
- The proof of $f \not\leq_{\mathrm{aW}} g$ is similar.

Definition (The powerspace $\mathsf{P}(\mathsf{Y})$)

For an assembly Y we define $\mathsf{P}(\mathsf{Y})$ by:

- Underlying set: $\{M \subseteq \mathsf{Y} \mid M \neq \emptyset\}$
- Multi-representation: $\delta_{\mathsf{P}(\mathsf{Y})}[q] \ni M \;:\Longleftrightarrow\; \delta_{\mathsf{Y}}[q] \cap M \neq \emptyset$.

Remark

$\delta_{\mathsf{P}(\mathsf{Y})}$ need not be admissible, even if δ_{Y} is.

Definition
For $f :\subseteq X \rightrightarrows Y$, define the function $f^{\mathsf{P}} : X \dashrightarrow \mathsf{P}(Y)$ by

$$f^{\mathsf{P}}(x) := f[x] \quad \text{for all } x \in \mathrm{dom}(f).$$

Proposition

- $f \equiv_{\mathrm{rW}} f^{\mathsf{P}}$
- $f \equiv_{\mathrm{srW}} f^{\mathsf{P}}$
- $f \equiv_{\mathrm{aW}} f^{\mathsf{P}}$
- $f \leq_{\mathrm{saW}} f^{\mathsf{P}}$, but not necessarily $f^{\mathsf{P}} \leq_{\mathrm{saW}} f$ (not even topologically).

Remark

$f \equiv_{\square} g :\Longleftrightarrow f \leq_{\square} g \leq_{\square} f$.

Definition

For a realizable problem f, the equivalence classes

(1) $\{g \mid g \equiv_{\mathrm{rW}} f\}$

(2) $\{g \mid g \equiv_{\mathrm{srW}} f\}$

(3) $\{g \mid g \equiv_{\mathrm{aW}} f\}$

(4) $\{g \mid g \equiv_{\mathrm{saW}} f\}$

are called the *entended Weihrauch degrees* of f induced by $\leq_{\mathrm{rW}}, \leq_{\mathrm{srW}}, \leq_{\mathrm{aW}}, \leq_{\mathrm{saW}}$, respectively.

Corollary

The (non-empty) extended Weihrauch degrees of types (1), (2), (3) contain a single-valued problem.

Recap

Classical Weihrauch degrees need not contain a single-valued problem.

Some Open Questions

- Do
  - the realizer-based reducibilities $\leq_{\mathrm{rW}}$, $\leq_{\mathrm{srW}}$

  or
  - the abstract reducibilities $\leq_{\mathrm{aW}}$, $\leq_{\mathrm{saW}}$

  yield a better generalisation of the classical Weihrauch reducibilities $\leq_{\mathrm{W}}$, $\leq_{\mathrm{sW}}$?
- Do (non-empty) entended Weihrauch degrees contain a function between *admissibly* multi-represented spaces?
- How do the topological versions behave?
  - Does $\leq_{\mathrm{rW}}^{\mathrm{top}} \neq \leq_{\mathrm{aW}}^{\mathrm{top}}$ hold as well?
- Do the extended Weihrauch reducibilities yield better insights than the classical ones?

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