Computably ANRs

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A compact metrizable space X has strong computable type if every copy of X in the Hilbert cube, which is semicomputable relative to some oracle O, is computable relative to O (see [2]). This is a generalization of the notion of computable type studied by Miller, Iljazovic et al, Hoyrup and the author (see for instance [9, 8, 7, 4, 1, 2, 3]).

It is proved in [2] that a necessary condition for a compact space X to have strong computable type is to satisfy the ϵ-surjection property for some $\epsilon > 0$, namely, every continuous function from X to itself which is ϵ-close to the identity must be surjective. This notion was studied in details in [3].

In this talk, we introduce the notion of computably ANR spaces and prove that for such spaces satisfying the ϵ-surjection property is sufficient to have strong computable type.

Let us give more details.

Let X be a compact metrizable space and $A \subseteq X$ be a compact subset. A retraction $r : X \to A$ is a continuous function such that $r|_A = \mathrm{id}_A$.

We define the notion of computably ANRs which is an effective version of absolute neighborhood retracts (ANRs), for more details about this class of spaces see [6] and [10].

Definition. A compact metrizable space X is a **computably absolute neighborhood retract (computably ANR)** if there exists a copy X_0 of X in the Hilbert cube and a computable retraction $r : U \to X_0$ for some open set U containing X_0.

Note that the definition implies that the copy X_0 is computable.

It is easy to see that an n-dimensional sphere is a computably ANR. More generally, one can prove that a finite simplicial complex is a computably ANR.

An interesting question is whether every compact manifold is a computably ANR.

Now, let us state our main result.

Theorem. *Let X be a computably ANR, if X satisfies the ϵ-surjection property then it has strong computable type.*

(Strong) computable type is already studied for ANRs such as finite simplicial complexes and compact manifolds, see [8, 1, 3]. Using the above theorem, it is not difficult to create new ANRs which have strong computable type.

The result above extends to compact pairs (X, B), but more assumptions are used, namely B being an ANR as well.

We note that Collins proved that the homology groups of computably ANRs are computable (see [5]).

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