Characterisations of polynomial-time and -space complexity classes over the reals

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Many recent works study how analogue models work, compared to classical digital ones ([6]). By "analogue" models of computation, we mean computing over continuous quantities, while "digital" models work on discrete structures, like bits. This led to a broader use of Ordinary Differential Equation (ODE) in computability theory. An interesting side-effect was the resolution of several open problems in computability and computable analysis, such as the existence of a universal ODE in [5] or various hardness problems related to dynamical systems in [9]. When talking about ODEs, especially in computer science, it is necessary to clarify what type of equations we consider: discrete or continuous. For example, in [5], it is shown that Turing machines can be simulated through continuous ODEs. We will use both formalisms.

From this point of view, the field of *implicit complexity* has also been widely studied and developed. It lies between logic and the theory of programming. For example, it allows one to write alternative characterisations of several computability and complexity classes, e.g. based on safe recursion [1], without any reference to the notion of machine. In the context of implicit complexity a standard notation for denoting the smallest class containing functions $f_1, ..., f_n$ and operators $op_1, ..., op_m$ is $[f_1, ..., f_n; op_1, ..., op_m]$.

Using these frameworks, we proved, using arguments from computable analysis, in [2] and [3] that we can algebraically characterise **P** and **PSPACE**, using discrete ODEs and more precisely the so-called Linear Length ODEs schemata.

We did so for **P**. We started by characterising the set of real numbers computable in polynomial time and of real sequences computable in polynomial time ([2]):

$$\overline{\mathbb{LDL}^{\bullet}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\mathrm{cond}}(x), \frac{x}{2}; \mathit{composition}, \mathit{linear\ length\ ODE}, \mathit{ELim}],$$

where π_i^k is the projection of the i^e coordinate of a k-vector, $\ell(x)$ is the size of the binary representation of an integer x, $\overline{\mathrm{cond}}(x)$ valuing 1 for $x > \frac{3}{4}$ and 0 for $x < \frac{1}{4}$, and *ELim* stands for "Effective Limit", that enable us to compute limits and make approximations. Thus we have $\overline{\mathbb{LDL}^{\bullet}} \cap \mathbb{R}^{\mathbb{N}} = \mathbf{P} \cap \mathbb{R}^{\mathbb{N}}$. Then, we moved to the characterisation of functions over the reals. But we cannot apply the same techniques for the computable functions in $\mathbb{R}^{\mathbb{R}}$. The problem comes from the fact that, in our settings, we can no longer guarantee the continuity of the functions and the computations in our contructions.

Thus it was necessary to have a different approach and continuously approximate some discrete functions that are mandatory for the proofs. The following algebra characterises the set of functions over the reals computable in polynomial time in [3]:

$$\overline{\mathbb{LDL}^{\circ}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3}; \mathit{composition}, \mathit{linear\ length\ ODE}, \mathit{ELim}],$$

where tanh stands for the hyperbolic tangent. This gives $\mathbb{LDL}^{\circ} \cap \mathbb{R}^{\mathbb{R}} = \mathbf{P} \cap \mathbb{R}^{\mathbb{R}}$. It turns out that we also characterised **PSPACE** using a similar algebra. For that, we introduced the notion of "robust" linear ODE, which means that the Linear ODEs are now assumed numerically stable. And we have the class $\overline{\mathbb{RLD}^{\circ}}$ characterising **PSPACE** ([3]), and $\overline{\mathbb{RLD}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \mathbf{PSPACE} \cap \mathbb{R}^{\mathbb{R}}$.

A natural extension of the above works is to use continuous ODEs as operators instead of discrete ODEs. To do so, we are inspired by [8], [4], where the use of continuous ODEs for such purpose has been developed. The major issues in that framework are the numerical stability and the management of the approximation errors. In order to link this characterisation to the previous one, we use a a construction due to [7], often called "Branicky trick", to replace discrete settings by continuous ones, thus the need to deal with an error bound. And, to have an equivalence, we need a proper notion of numerical stability, so the ODEs are stable under discretisation. We will present some original results on these aspects.

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