Discontinuous IVPs with unique solutions

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The study of ordinary differential equations (ODEs) and initial value problems (IVPs) with discontinuous right-hand terms has many applications to a wide range of problems in mechanics, electrical engineering and theory of authomatic control. Broadly speaking, discontinuous ODEs of the form $y'(t) = f(t, y)$ can be divided into two main categories [5]: one in which f is continuous in y for almost all t and one in which f is discontinuous on an arbitrary subset of its domain. In the first case, existence and unicity for solutions of the IVPs can be discussed under specific requirements on f, such as the Carathéodory conditions [1]. In the second case, the most common approach is to study the dynamic using differential inclusions of the form $y'(t) \in F(t, y)$ by identifying the correct definition of F on the set of discontinuity points. In both cases, the solution, when unique, is an absolutely continuous function y such that $y'(t) = f(t, y)$ is defined almost everywhere in an interval I.

We choose to analyze a different scenario: discontinuous IVPs for which the solution is necessarily unique and the equation $y'(t) = f(t, y)$ is defined everywhere on I. In other words, we assume existence and unicity and we focus on finding an analytical procedure to obtain such solution from f and the initial condition. In this sense, the point of view is similar to the one in [2], where it is shown that when y is unique, then it is computable if the IVP is. Nonetheless, unicity when f is discontinuous might imply noncomputability of y even when the set of discontinuity points is trivial.

We first demonstrate the difficulty by means of an example: we construct a bidimensional IVP such that, despite having computable initial condition and f computable everywhere except a straight line, has a solution that assumes a noncomputable value at a fixed integer time. This demonstrates the capability of these dynamical systems of generating highly complicated solutions even when the structure of the discontinuity points in the domain is particularly simple. Therefore, our goal is to characterize the definability of y, generalizing the result obtained in [2] for computability to a wider class of IVPs with unique solutions.

Our approach is close in spirit to that of A. Denjoy, who provided with his totalization method an extension to the Lebesgue integral in order to generalize the operation of antidifferentiation to a wider class of derivatives [3]. This perspective fits within a wider research field that explores set theoretical descriptions of the complexity of operations such as differentiation and integration. For example, [8] makes use of the notion of differentiable rank from [6] to present a precise lightface classification of differentiable functions based on how complex their derivative can be. In [4],[7] instead the authors conduct similar treatments for antidifferentiation.

More formally, the dynamical systems we are considering are IVPs of the form: given an interval $[a, b]$, a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$, a point $y_0 \in E$ and a function

$f : E \to E$ we have:

$$\begin{cases} y'(t) = f(y(t)) \\ y(a) = y_0 \end{cases} \tag{1}$$

for some $y : [a,b] \to \mathbb{R}^r$ with $y([a,b]) \subset E$. Under the assumption of unicity of y we show that two conditions on the right-hand term f are sufficient for obtaining the solution via transfinite recursion up to a countable ordinal. In order to do so we make use of a construction inspired by the Cantor-Bendixson analysis where the set derivative operator is replaced by the action of excluding the set of discontinuity points of f. More precisely, we define:

Definition 1 *Consider a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to \mathbb{R}^r$. Let $\{E_\alpha\}_{\alpha<\omega_1}$ and $\{f_\alpha\}_{\alpha<\omega_1}$ be transfinite sequences such that $f_\alpha = f \restriction_{E_\alpha}: E_\alpha \to \mathbb{R}^r$ defined as following: let $E_0 = E$; for all $\alpha = \beta + 1$ let $E_\alpha = \{x \in E_\beta : f_\beta$ is discontinuous in $x\}$; for all α limit ordinal let E_α be $E_\alpha = \cap_\beta E_\beta$ with $\beta < \alpha$. We call $\{E_\alpha\}_{\alpha<\omega_1}$ the sequence of f-removed sets on E.*

Studying the structure of the sequence of f-removed sets on E allows us to prove that the two following conditions on f are sufficient for obtaining the solution. The conditions are: 1) f is a function of class Baire one 2) For all closed $K \subseteq E$ the set of discontinuity points of $f \restriction_K$ is a closed set. This result is obtained with our main theorem:

Theorem 2 *Consider a closed interval, a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to E$ such that, given an initial condition, the IVP of the form of Equation 1 with right-hand term f has a unique solution on the interval. If f is a function of class Baire one such that for every closed $K \subseteq E$ the set of discontinuity points of $f \restriction_K$ is closed, then we can obtain the solution analytically via transfinite recursion up to an ordinal α such that $\alpha < \omega_1$.*

This result expresses the analogy between this context and the totalization method in [3], leaving open the possibility of defining a rank for the IVPs related to constructible ordinals in order to populate a hierarchy similar to the one presented in [8] for differentiable functions.

References

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