# Benign Approximations and Non-Speedability*

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A real number is called *left-computable* if there exists a computable non-decreasing (or, equivalently, increasing) sequence of rational numbers converging to it. A real number $x$ is called *computable* if there exists a computable sequence $(x_n)_n$ of rational numbers satisfying $|x - x_n| < 2^{-n}$ for all $n \in \mathbb{N}$. It is easy to see that every computable number is left-computable, but the converse is not true. It is also easy to see that a real number $x$ is computable if and only if there exists a computable non-decreasing sequence $(x_n)_n$ of rational numbers with $x - x_n < 2^{-n}$ for all $n \in \mathbb{N}$. However, if we only require that the condition $x - x_n < 2^{-n}$ be satisfied for infinitely many $n \in \mathbb{N}$, we obtain a different subset of the left-computable numbers introduced by Hertling, Hölzl, and Janicki [5].

**Definition 1** (Hertling, Hölzl, Janicki [5])**.** *A real number $x$ is called* regainingly approximable *if there exists a computable non-decreasing sequence of rational numbers $(x_n)_n$ converging to $x$ with $x - x_n < 2^{-n}$ for infinitely many $n \in \mathbb{N}$.*

Obviously, every computable number is regainingly approximable, but the converse is not true [5]. Thus, with respect to inclusion, the regainingly approximable numbers are properly lodged between the computable and the left-computable numbers.

Another approach to define an interesting subset of the real numbers was introduced by Hertling and Janicki [6]. They used the concept of computable convergence.

**Definition 2** (Hertling, Janicki [6])**.**

*(1) A sequence $(x_n)_n$ is called* nearly computably convergent *if it converges and, for every computable increasing function $s\colon \mathbb{N} \to \mathbb{N}$, the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges computably to 0.*

*(2) A real number is called* nearly computable *if there exists a computable sequence of rational numbers which converges nearly computably to it.*

Naturally, every computable number is nearly computable, but the converse is not true by a theorem of Downey and LaForte [4, Theorem 3] combined with results of Stephan and Wu [10, Theorem 5] and Hertling and Janicki [6, Proposition 7.2].

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In this article we are interested in the relationship between the set of left-computable nearly computable numbers and that of regainingly approximable numbers; as every computable number is both nearly computable and regainingly approximable, we are only interested in non-computable numbers. It turns out that both subsets are incomparable under inclusion, but they have a non-trivial intersection.

**Theorem 3.** *There exists a regainingly approximable number which is nearly computable, but not computable.*

**Theorem 4.** *There exists a left-computable number which is nearly computable but not regainingly approximable.*

Both examples are built using infinite injury priority constructions. Finally, there are also regainingly approximable numbers which are not nearly computable.

For $A \subseteq \mathbb{N}$, we define a real number in the interval $[0, 1]$ via $x_A := \sum_{n \in A} 2^{-(n+1)}$. Then clearly $A$ is computable if and only if $x_A$ is computable. If $A$ is only assumed to be computably enumerable, then $x_A$ is a left-computable number; the converse is not true as pointed out by Jockusch (see Soare [9]). We say that a real number $x \in [0, 1]$ is *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x_A = x$. A result of Downey, Hirschfeldt, and LaForte [3, Theorem 2.15] implies that every strongly left-computable number that is nearly computable is, in fact, computable. But Hertling, Hölzl, and Janicki [5] established the existence of regainingly approximable numbers that are strongly left-computable without being computable. Such a number can therefore not be nearly computable.

With the help of Theorem 4 we obtain a negative answer to an open problem stated by Merkle and Titov [7]; they investigated when and to what extent computable approximations to left-computable numbers can be accelerated.

**Definition 5** (Merkle, Titov [7])**.** *A left-computable number $x$ is called* speedable *if there exists a constant $\rho \in \,]0, 1[$ and a computable increasing sequence of rational numbers $(x_n)_n$ converging to $x$ such that there are infinitely many $n \in \mathbb{N}$ with $\frac{x - x_{n+1}}{x - x_n} \leq \rho$.*

Among other results, they gave a direct proof that speedable numbers are not Martin-Löf random[1] and asked whether the inverse implication holds as well; that is, whether a left-computable number is speedable if and only if it is not Martin-Löf random. It turns out that the study of regaining approximability and nearly computability is an approach for solving this problem.

**Proposition 6.** *Every regainingly approximable number is speedable.*

However, the converse is not true. Merkle and Titov [7] showed that every strongly left-computable number is speedable. Hertling, Hölzl and Janicki [5] showed that there exists a strongly left-computable number that is not regainingly approximable. Combining these two results together, there exists a speedable number that is not regainingly approximable.

Thus, in general, the regainingly approximable numbers are a proper subset of the speedable numbers. However, the two notions become equivalent once we restrict ourselves to nearly computable numbers.

[1] This was also implicitly shown by Barmpalias and Lewis-Pye [1, Theorem 1.7]. For general background on Martin-Löf random numbers, refer to Downey and Hirschfeldt [2] or Nies [8].

**Theorem 7.** *Let $x$ be a left-computable number that is nearly computable. Then the following statements are equivalent:*

*(1) $x$ is regainingly approximable.*

*(2) $x$ is speedable.* □

Combining Theorem 7 with Theorem 4, we obtain as a corollary that there exists a left-computable number which is nearly computable but not speedable. Stephan and Wu [10] showed that a left-computable number which is nearly computable cannot be Martin-Löf random. This implies our final main result, a negative answer to the question of Merkle and Titov [7].

**Corollary 8.** *There exists a left-computable number which is not speedable and not Martin-Löf random.* □

# References


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