REORDERED COMPUTABLE NUMBERS

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Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function which tends to infinity. We can define the function $u_f\colon \mathbb{N} \to \mathbb{N}$ by $u_f(n) := |\{k \in \mathbb{N} \mid f(k) = n\}|$ for all $n \in \mathbb{N}$. Then, for every bijective function $\sigma\colon \mathbb{N} \to \mathbb{N}$ the composition $f \circ \sigma$ also tends to infinity and we have $u_{f\circ\sigma} = u_f$. It is also easy to see that there exists a unique non-decreasing and unbounded function $f^*\colon \mathbb{N} \to \mathbb{N}$ with $u_{f^*} = u_f$. Note that, in general, both u_f and f^* are not computable even if f is computable.

Let $(x_n)_n$ be a convergent sequence of real numbers and let $x := \lim_{n\to\infty} x_n$. A function $s\colon \mathbb{N} \to \mathbb{N}$ is called a *modulus of convergence* for $(x_n)_n$ if for all $n \in \mathbb{N}$ and for all $i > s(n)$ we have $|x - x_i| \leq 2^{-n}$. If such a sequence has a computable modulus of convergence, we say that it *converges computably.* A real number is called *computable* if there exists a computable sequence of rational numbers converging computably to it. It is called *left-computable* if there exists a computable increasing sequence of rational numbers converging to it. It is well-known that every computable number is left-computable but not vice-versa. For a set $A \subseteq \mathbb{N}$, we define a real number in the interval $[0, 2]$ via $x_A := \sum_{n\in A} 2^{-n}$. Then clearly A is computable if and only if x_A is computable. If A is computably enumerable, then x_A is a left-computable number, but the converse is not true. A real number $x \in [0, 2]$ is called *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x_A = x$. For the remainder of this article we will only consider positive real numbers.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function. Then $\left(\sum_{k=0}^{n} 2^{-f(k)}\right)_n$ is an increasing sequence of rational numbers. If f is computable, then this sequence is computable as well. If, in addition, the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges, then the limit is a left-computable number. It is easy to see that for every left-computable number there exists a computable function $g\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=0}^{\infty} 2^{-g(k)}$ converges to it. In this article we are interested in functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges. Of course, this also implies that f tends to infinity. From classical analysis we know that for every bijective function $\sigma\colon \mathbb{N} \to \mathbb{N}$ the rearranged series $\sum_{k=0}^{\infty} 2^{-f(\sigma(k))}$ also converges, toward the same limit. However, the speed of convergence of the rearranged series may differ from the original series. Clearly, a rearranged series converges fastest if its biggest jumps are made first.

Proposition 1. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges.*

1. *If $s\colon \mathbb{N} \to \mathbb{N}$ is a modulus of convergence, then s is also a modulus of convergence for the reordered series $\sum_{k=0}^{\infty} 2^{-f^*(k)}$.*
2. *The following are equivalent:*
   - *The series $\sum_{k=0}^{\infty} 2^{-f^*(k)}$ converges computably.*
   - *The series $\sum_{k=0}^{\infty} u_f(k) \cdot 2^{-k}$ converges computably.*

Definition 2. *We say that a real number number x is* reordered computable *if there exists a computable function $f\colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$ such that the reordered series $\sum_{k=0}^{\infty} 2^{-f^*(k)}$ converges computably.*

By definition, every reordered computable number is a left-computable number. We give a simple example for a reordered computable number. If $x \in\,]0, 2]$ is a strongly left-computable

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number, then there exists a computable injective function $f\colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$. Since f is injective, we have $u_f(n) \leq 1$ for all $n \in \mathbb{N}$, and the reordered series $\sum_{k=0}^{\infty} u_f(k) \cdot 2^{-k}$ converges computably. Therefore, every computable number is reordered computable, and there are reordered computable numbers which are not computable. Other, more abstract examples can be obtained through the next two theorems.

Theorem 3. *Let x be a left-computable number such that there exists a computable function $f\colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$ and $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} < 2$. Then x is reordered computable.*

Theorem 4. *Let $A \subseteq \mathbb{N}$ be an infinite set such that the number x_A is left-computable. If A is not immune, then x_A is reordered computable.*

It is not difficult to show that the reordered computable numbers are closed under addition and multiplication. Furthermore, they are closed downwards under the Solovay reduction.

Theorem 5. *Let x and y be left-computable numbers with $x \leq_S y$. If y is reordered computable, then x is reordered computable as well.*

Looking for counterexamples for reordered computable numbers, one can find them in the set of left-computable numbers which are nearly computable. Hertling and Janicki [3] define a real number x to be *nearly computable* if there exists a computable sequence of rational numbers $(x_n)_n$ converging to x such that for every computable increasing function $s\colon \mathbb{N} \to \mathbb{N}$, the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges computably to 0. Naturally, every computable number is both left-computable and nearly computable, but the converse is not true by a theorem of Downey and LaForte [1, Theorem 3] combined with results of Stephan and Wu [4, Theorem 5] and Hertling and Janicki [3, Proposition 7.2].

Proposition 6. *Let x be a reordered computable number which is nearly computable. Then x is even computable.*

Other counterexamples are left-computable numbers which are Martin-Löf random. Let Ω be such a number. It is well-known that for every left-computable number x we have $x \leq_S \Omega$. Since the reordered computable numbers are closed downwards under the Solovay reduction and not every left-computable number is reordered computable, Ω is not reordered computable as well. However, we can even prove a stronger result[1], namely that every reordered computable number can split into the sum of two regainingly approximable numbers. Hertling, Hölzl and Janicki [2] define a real number x to be *regainingly approximable* if there exists a computable increasing sequence of rational numbers $(x_n)_n$ converging to x with $x - x_n \leq 2^{-n}$ for infinitely many $n \in \mathbb{N}$. Obviously, every regainingly approximable number is left-computable and every computable number is regainingly approximable, but none of these implications can be reversed [2].

Theorem 7. *Let x be a reordered computable number. Then there exist regainingly approximable numbers α and β with $\alpha + \beta = x$.*

References

[1] R. Downey and G. LaForte. Presentations of computably enumerable reals. *Theoretical Computer Science*, 284(2):539–555, 2002.

[2] P. Hertling, R. Hölzl, and P. Janicki. Regainingly approximable numbers and sets, 2023. Available at https://arxiv.org/abs/2301.03285.

[3] P. Hertling and P. Janicki. Nearly computable real numbers, 2023. Available at https://arxiv.org/abs/2301.08124.

[4] F. Stephan and G. Wu. Presentations of K-trivial reals and Kolmogorov complexity. In B. Cooper, B. Löwe, and L. Torenvliet, editors, *Proceedings of the 1st International Conference on Computability in Europe*, Lecture Notes in Computer Science 3526, pages 461–469. Springer, 2005.

[1] This result is also a strengthening of the splitting theorem by Hertling, Hölzl and Janicki [2, Theorem 37].