1 is definable in the Weihrauch degrees

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We explore structural properties of the Weihrauch degrees (see [1] for context) as a partial order. By 1 we denote the Weihrauch degree of the identity on Baire space. This Weihrauch degree contains exactly the computable multivalued functions with a computable point in their domain. We show that 1 has a special position in the partial order of the Weihrauch degrees.

Theorem 1. The Weihrauch degrees above 1 are dense, i.e. for any $1 \leq_{\mathrm{W}} \mathbf{a} <_{\mathrm{W}} \mathbf{b}$ there is a $\mathbf{c}$ with $\mathbf{a} <_{\mathrm{W}} \mathbf{c} <_{\mathrm{W}} \mathbf{b}$. On the other hand, if $1 \nleq_{\mathrm{W}} \mathbf{a}$, then there are $\mathbf{b} \geq_{\mathrm{W}} \mathbf{a}$ and $\mathbf{c} >_{\mathrm{W}} \mathbf{b}$ such that there is no $\mathbf{d}$ with $\mathbf{b} <_{\mathrm{W}} \mathbf{d} <_{\mathrm{W}} \mathbf{c}$.

There is a second distinguished property of 1. We recall that that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a} <_{\mathrm{W}} \mathbf{b}$ if whenever $\mathbf{c} <_{\mathrm{W}} \mathbf{b}$, then already $\mathbf{c} \leq_{\mathrm{W}} \mathbf{a}$.

Theorem 2. If $\mathbf{b}$ is a strong minimal cover of some Weihrauch degree, then $\mathbf{b} \leq_{\mathrm{W}} 1$. Moroever, 1 is a strong minimal cover of the identity restricted to the non-computable elements in Baire space.

Either one of our two theorems implies the following:

Corollary 3. 1 is definable in the partial order of the Weihrauch degrees.

As the lower cone of 1 in the Weihrauch degrees is isomorphic to the dual of the Medvedev degrees [2], combining the preceding corollary with a theorem on the first-order theory of the Medvedev degrees obtained independently by Shafer [5] and by Lewis, Nies and Sorbi [3] then yields the following:

Corollary 4. The first-order theory of the Weihrauch degrees is recursively isomorphic to third-order true arithmetic.

Our proofs of Theorems 1 and 2 establish more than the corresponding statements. In fact, we obtain a precise classification of the strong minimal covers and the empty intervals in the Weihrauch degrees.

Proving the definability of 1 in the partial order of the Weihrauch degrees is only scratching the surface of definability questions in the hithertho-studied structure of the Weihrauch degrees, as remarked briefly in [4].

References

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