Decision Problems in Analysis

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One of the staples of theoretical computer science is the decision problem: For a fixed set $S \subseteq \mathbb{N}$, determine whether a given number $n \in \mathbb{N}$ belongs to S. This definition does not generalise well to continuous data structures – connected computable metric spaces do not admit any non-trivial decidable subsets. This simple observation has prompted a long line of investigations into possible definitions of computability [1, 3, 4] and complexity [2, 10, 5] for subsets of more general metric spaces.

In this talk, I will discuss a natural generalisation of decidability from $\mathbb{N}$ to arbitrary computable metric spaces that appears to have been less thoroughly explored so far: Call a subset $S \subseteq X$ of a computable metric space X *maximally partially decidable* [7] if there exists an algorithm which takes as input a point $x \in X$, halts if $x \in X \setminus \partial S$, and upon halting correctly reports whether x is contained in S. This amounts to computing the best continuous approximation [6] of the characteristic function of S. Call S *maximally partially decidable in polynomial time* if the algorithm halts within a number of steps that is controlled polynomially in the size of the input and the negative logarithm of its distance to ∂S.

I will compare the above definitions to existing notions of computability and complexity for closed sets and present a few recent applications to verification problems for dynamical systems [8, 9].

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