LENGTH FUNCTIONS AND THE DIMENSION OF POINTS IN SELF-SIMILAR FRACTAL TREES

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The tools from algorithmic information theory, particularly the notion of effective dimension of an individual point, have found successful application in the study of fractal geometry. As a particularly striking example, Lutz and Mayordomo in [LM08] provide a general formula for calculating the effective dimension $\dim(x)$ of a point x in a self-similar fractal F in $\mathbb{R}^n$:

$$\dim(x) = \mathrm{sdim}(F)\dim^{\mu}(y),$$

where $\mathrm{sdim}(F)$ is the similarity dimension of F, $\dim^{\mu}(\cdot)$ is an effective analogue of the Billingsley dimension with respect to a specific probability measure μ defined in terms of the fractal F, and y is an infinite sequence over some finite alphabet that serves as a code for x as an element of F. They further established an analogous for effective strong dimension $\mathrm{Dim}(x)$ of points $x \in F$.

In this talk, I will discuss similar results for points in infinite self-similar trees over some finite alphabet [Por23]. Our main result relies on the machinery of coding with unequal costs from information theory, which in this context, amounts to considering notions of algorithmic information theory in terms of length functions that do not necessarily measure the length of a string of symbols merely in terms of the number of symbols in the string (see, e.g. [Var71, Cha86, Abr94, Abr97, GKY02]). More generally, for a given m-ary tree T for some $m \in \mathbb{N}$, we identify the relationship between the following quantities from seemingly disparate areas:

(1) α, the channel capacity of a code with unequal letter costs that is defined in terms of T,
(2) $\mathrm{sdim}(F)$, the similarity dimension of an infinite self-similar tree F generated by T, and
(3) $\log(\lambda)$, the negative logarithm of the Perron eigenvalue of the adjacency matrix of a specific directed graph G determined by T,

are given by the equalities

$$\alpha = \log(m) \cdot \mathrm{sdim}(F) = \log(\lambda)$$

(where the logarithm here is taken to the base 2). In the case that T is a binary tree, we get the equality of the three quantities (1)-(3) listed above. From the relationship between (1) and (2), we derive our analogue of the above-mentioned Lutz/Mayordomo result that holds for points in self-similar trees. Using (3), we show how the measure μ used in the calculation of the effective version of Billingsley dimension mentioned above can be obtained from a transformation of the unique measure of maximal entropy on a specific subshift related to the tree T (in this case, the Parry measure on a specific sofic shift).

I will further discuss follow-up work with Adam Case (currently in progress), in which we apply the analogue of the Lutz/Mayordomo formulas to study the behavior of the dimension of sequences transformed by morphisms (here, a morphism on the alphabet $\Sigma_k = \{0, \ldots, k-1\}$ is simply a map from Σ_k to Σ_k^*). In particular, by combining results from [Por23] with a divergence formula for effective dimension established by Lutz [Lut11], we obtain an elegant formula expressing the relationship between the effective dimension of a point given as input to a map determined by a morphism and the effective dimension of the output (for a sufficiently broad class of points and morphisms); a similar result for strong dimension holds as well. Finally, I will also briefly discuss extensions of the results from [Por23], as well as the more above-mentioned results on morphisms, in the context of finite-state dimension.

References

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