Exploring the abyss in Kleene computability theory

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Abstract: I will present some results from my ongoing project with Dag Normann on the computational properties of the uncountable, namely a new discovery in Kleene's computability theory based on S1-S9 ([1]) and our recent equivalent λ-calculus formulation based on fixed point operators ([2]).

In a nutshell, I will present basic operations stemming from mainstream mathematics that are computable in Kleene's quantifier $\exists^2$, while *slight* mathematical variations are computable in Kleene's quantifier $\exists^3$ but not in weaker oracles ([3]). Since the difference between $\exists^2$ and $\exists^3$ is considerable, we say that these two classes are *separated by an abyss.*

First of all, Kleene's S1-S9 provide a formal framework for the notion

the object X is computable in the object Y

where X, Y are objects of any finite type. In case X, Y are real numbers, S1-S9-computability reduces to Turing computability, i.e. the former is an extension of the latter. While S1-S8 only formalise a basic form of primitive recursion, the schema S9 is rather ad hoc in nature, hard-coding as it does the recursion theorem for S1-S9-computability theory. By contrast, the recursion theorem for Turing machines is derived from first principles in [4].

In light of the previous, it is a natural question whether there is a (more) conceptually pleasing formulation of S1-S9, which is one of the topics of [2]. In particular, it is shown that S1-S9 can be equivalently captured by the computational model consisting of the following extended lambda calculus:

- Kleene's schemas S1-S8 capturing primitive recursion,
- λ-abstraction for all finite types,
- the least fixed point operators μ^σ for all finite types σ.

The operator in the final item is such that $\mu^\sigma x.s(x)$ is the least fixed point of the $\sigma \to \sigma$-mapping $\lambda x.s(x)$ which is assumed *monotone* as follows:

$x^\sigma \preceq_\sigma y^\sigma$ means that the graph of y is included in the graph of x and

the mapping $\lambda x.s(x)$ is *monotone* if $(\forall x^\sigma, y^\sigma)(x \preceq_\sigma y \to s(x) \preceq_\sigma s(y))$.

We introduce this new model and briefly sketch its advantages over S1-S9.

Secondly, we identify a number of basic operations (e.g. finding suprema or points of continuity) stemming from mainstream mathematics that are 'close to computable' in that they are computable from $\exists^2$ (or weaker oracles). Perhaps surprisingly, *slight* variations of these basic operations (like extending the function class) are computable in $\exists^3$ but not in weaker oracles. The following examples give the reader an idea of this abyss.

The functional $\exists^2$ *can compute* $\sup_{x\in[p,q]} f(x)$ *for* $f : [0,1] \to \mathbb{R}$ *which is either cadlag, Baire 1, or quasi-continuous.*

The functional $\exists^3$ *can compute* $\sup_{x\in[p,q]} f(x)$ *for* $f : [0,1] \to \mathbb{R}$ *which is either regulated, Baire 2, or cliquish, but no weaker oracle suffices.*

Mathematically speaking, the operations on the first and second line are close as respectively cadlag and regulated, Baire 1 and Baire 2, and quasi-continuous and cliquish, are closely related function classes in real analysis. We discuss similar results for other function classes and operations.

An important observation concerning the abyss is that the operations computable in $\exists^2$ involve function classes for which $f(x)$ for any $\in [0,1]$ can always be approximated using $f(q)$ for $q \in [0,1] \cap$
mathbbQ. The other side of the abyss lacks this approximation device.

References

[1] Stephen C. Kleene, *Recursive functionals and quantifiers of finite types. I*, Trans. Amer. Math. Soc. **91** (1959), 1–52.

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[4] Robert I. Soare, *Recursively enumerable sets and degrees*, Perspectives in Mathematical Logic, Springer, 1987.