Computing signed distance functions

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We give a new conceptual proof of K. Weihrauch's recent result that two crossing computable curves in the unit square have a computable intersection point (cf. [7]). The key idea of the new proof is to compute *signed distance functions* for closed subsets of the unit square (see Theorem 1). The corresponding algorithm is based on computing winding numbers. As our second result we show that the latter forms a computable operator (see Theorem 2). We assume some familiarity with Computable Analysis (cf. [2, 4, 5, 6]).

Signed distance functions

For any closed subset A of a metric space X, the distance function for A defined by $d_A(x) := \inf_{a \in A} d(a, x)$, where d is the metric of X, is continuous. It is well-known in Computable Analysis that any closed subset of the Euclidean space $\mathbb{R}^k$ can be computably transformed into its distance function (cf. [1]).

By a *signed distance function* for A we mean a continuous function $h\colon X \to \mathbb{R}$ such that $x \mapsto |h(x)|$ is the distance function for A.

Clearly, any continuous function $\gamma\colon [0;1] \to [0;1]^2$ such that $\gamma(0) = (0,0)$, $\gamma(1) = (1,1)$ separates the corner points $(0,1)$ and $(1,0)$ in the sense that any continuous path inside the unit square connecting these corner points intersects $\mathrm{image}(\gamma)$. Our idea is to construct a signed distance function h for the curve $\mathrm{image}(\gamma)$ that reflects this phenomen by mapping $(0,1)$ and $(1,0)$ to numbers with opposite signs, whenever they are not on the curve.

Theorem 1 *There is a computable function* SD *that maps any continuous function* $\gamma\colon [0;1] \to [0;1]^2$ *with* $\gamma(0) = (0,0)$ *and* $\gamma(1) = (1,1)$ *to a continuous function* $h\colon [0;1]^2 \to \mathbb{R}$ *satisfying:*

(1) $|h(z)| = d_{\mathrm{image}(\gamma)}(z)$ *for all* $z \in [0;1]^2$,

(2) $h(0,1) \leq 0$,

(3) $h(1,0) \geq 0$.

In order to achieve continuity of $\mathrm{SD}(\gamma)$, the algorithm for SD uses winding numbers to decide whether $\mathrm{SD}(\gamma)$ maps a point $z \notin \mathrm{image}(\gamma)$ to a positive or a negative real.

Winding numbers

In Complex Analysis, a *closed* path is a continuous function $\Gamma\colon [a;b] \to \mathbb{C}$ such that $\Gamma(a) = \Gamma(b)$. Given $z \in \mathbb{C} \setminus \mathrm{image}(\Gamma)$, the winding number $\omega(\Gamma, z)$ is defined as the complex line integral $\frac{1}{2i\pi} \int_\Gamma \frac{1}{\zeta - z}\, d\zeta$. The winding number is known to be always an integer (cf. [3]). It equals 0 if Γ does not orbit z, whereas it is equal to 1 or -1 if Γ orbits z exactly once. We stress that the line integral is well-defined and finite, even if Γ is not rectifiable, because the integrand $\zeta \mapsto \frac{1}{\zeta - z}$ is an holomorphic function on $\mathbb{C} \setminus \{z\}$.

Theorem 2 *The partial function mapping any continuous function* $\Gamma\colon [0;2] \to \mathbb{C}$ *with* $\Gamma(0) = \Gamma(2)$ *and any* $z \in \mathbb{C} \setminus \mathrm{image}(\Gamma)$ *to the winding number* $\omega(\Gamma, z)$ *is computable.*

Intersection points of computable planar curves

We use Theorem 1 to give a new conceptual proof of a recent result by K. Weihrauch stating that two computable functions $\phi, \psi\colon [0;1] \to [0;1]^2$ satisfying

$$\phi(0) = (0,0),\ \phi(1) = (1,1),\ \psi(0) = (0,1),\ \psi(1) = (1,0)$$

have a computable intersection point (cf. [7]). Any zero t of the real-valued function $\mathrm{SD}(\phi) \circ \psi$ has the property that $\psi(t)$ is an intersection point of ϕ, ψ. This observation establishes that the discontinuous problem IP of finding such an intersection point is ordinary Weihrauch reducible (cf. [2]) to the Intermediate Value Theorem IVT (a fact known from [8]). The fact $\mathsf{IP} \leq_{\mathrm{W}} \mathsf{IVT}$ shows not only the aformentioned result but also that IP restricted to instances (ϕ, ψ) that have at most countably many intersection points is computable (in contrast to the full problem IP).

References

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