Continuity and computability: Transition mechanism and admissible extension

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The main theme of this treatise is an epistemological consideration on some 'computable functions in an extended sense'. [1]

We will confine the universe of discourse to real numbers, real number sequences and real total functions throughout, and work with Euclidean topology on the real line. We assumse the traditional theory of computable (continuous) functions as well as some theories of computability in the extended sense. We have developed the theory of 'irrational-based (IB-) computability' in [2]. It is a mathematical framework which provides a unifying conception of computability for the functions we had treated.

The ground on which we accept IB-computability as a sound notion of computability may be explained as follows. The conventional computability of a function is an effectivization of 'locally uniform (called here LU-) continuity', while one could effectivize 'continuity' (without any modifier; called here 'location-wise (LW-) continuity'), resulting in 'LW-computability'. [2] IB-computability can be viewed as a natural extension of LW-computability. This way of thinking is supported by the theory of 'conjunctive extension' of a mathematical theory, which the first author has developed in [1] and other works.

One aim of the present study is to examine the transition process from continuity to computability, whose state will be called the 'transition mechanism'. Another is to deduce that extending LW-computability to IB-computability is an 'admissible (natural) extention'. ε will denote a rational number.

Definition 1 A function f is 'LW-computable' if there is an effective method, e, which performs the following tasks. (i) e computes $f(x)$ for every computable x. (ii) To each pair of computable x and $\varepsilon > 0$, e assigns a number $\delta = \delta_{x,\varepsilon} > 0$, so that it holds: (1) $\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. [3]

Definition 2 Let $\mathbf{I}$ denote the set of all irrational numbers. A function f is IB-continuous if $\forall x \in \mathbf{I} \forall \varepsilon > 0 \exists \delta > 0 \forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ holds.

Definition 3 ([2]) f is called 'IB-computable' if there is an effective method, e, which act as in Definition 1 for every computable $x \in \mathbf{I}$.

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[1]The unpublised full text is posted, under the same title, on the site https://independent.academia.edu/MarikoYasugi .

[2]Needless to say, LU-continuity and LW-continuity are classically equivalent, but their effectivizations manifest different phases.

[3]We usually define computability for a computable sequence of numbers.

Proposition 1) LW-continuity implies IB-continuity. Computability implies LW-computability, which in turn implies IB-computability. The implications are strict.

2) LW-computability is closed under certain properties.

Definition 4 ([1]) (1) The 'conjunctive principle' of concept extension of a mathematical theory consists of the three conditions $1°\sim 3°$: $1°$ Inner necessity (necessity from within a mathematical theory); $2°$ Substantial conservation (conservation of the content); $3°$ Formal conservation (conservation of the laws characterizing the original concepts).

(2) An extension of a mathematical theory is called 'admissible' if the reslutting theory satisfies the conjunctive principle.

Theorem IB-continuity is an admissible extension of LW-continuity, and IB-computability is an admissible extension of LW-computability.

Schema $f, A : \Xi \mapsto B; L$ will express that 'B is obtained from f and A by the method Ξ and satisfies the law L', where f is presupposed to be a total function. Ξ being void means that the method is not specified, and Ξ being e means that the method be effective. L may be empty. Let $\mathbf{X}$ denote a set of real numbers. Consider the schema (a)& (b): (a) $f, x \in \mathbf{X} : \Xi \mapsto f(x)$; (b) $f, x \in \mathbf{X}, \varepsilon(> 0), \Xi \mapsto \delta > 0; \forall y.|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. If $\mathbf{X}$ is the set of all real numbers, $\mathbf{R}$, and Ξ is void, then the schema (named α) asserts LW-continuity of f. If $\mathbf{X}$ is the set of computable real numbers, $\mathbf{C}$, and Ξ is e, then the schema (named β) asserts the LW-computability of f.

Definition 5 The transition from α to β is called the transition mechanism from LW-continuity to LW-computability. Similarly with IB-continuity and IB-computability with $\mathbf{X} = \mathbf{I}$ and $\mathbf{X} = \mathbf{I} \cap \mathbf{C}$.

This is neither extension nor restriction, but the transition to computability has emerged from within the continuity and, since the law L is common to α and β, the formal conservation holds. If, in β, $\mathbf{C}$ is replaced by $\mathbf{R}$ and e is removed, the result turns out to be α. In this sense, substantial conservation also holds.

References

[1] Mariko Yasugi, *Two types of concept extension in mathematics*, Dissertation (Kyoto University), 2014. (in Japanese) Available at: http://hdl.handle.net/2433/188398 DOI 10.14989/doctor.k17995

[2] Mariko Yasugi, Yoshiki Tsujii, Takakazu Mori, *Irrational-based computability of functions*, Advances in Mathematical Logic: SAML 2018, Kobe; Springer (2021), 181-204. Available also at https://independent.academia.edu/MarikoYasugi