Computably ANRs

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Hilbert cube

Definition

The Hilbert cube is the space $Q = [0,1]^{\mathbb{N}}$ endowed with the product topology, induced by the complete metric

$$d_Q(x,y) = \sum_i 2^{-i}|x_i - y_i|$$

where $x = (x_0, x_1, \dots)$ and $y = (y_0, y_1, \dots)$.

Some facts about the Hilbert cube

- Let $(\alpha_i)_{i\in\mathbb{N}}$ be a computable enumeration of the points of Q having rational coordinates, finitely many of them being non-zero, the space (Q, d_Q, α) is a computable metric space,

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- Every computable metric space embeds effectively into the Hilbert cube,
- There exists a dense sequence of computable functions $f_i : Q \to Q$ in the space of continuous functions from Q to Q.

Semicomputable/computable compact sets

Definition

A compact set X in $(Q, (B_i)_{i \in \mathbb{N}})$ is

1. Semicomputable if the set

$$\{(i_1, \ldots, i_n) \in \mathbb{N}^* : X \subseteq B_{i_1} \cup \ldots \cup B_{i_n}\}$$

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1. Computable if it is semicomputable and the set

$$\{i \in \mathbb{N} : X \cap B_i \neq \emptyset\}$$

is c.e.

Strong computable type

Definition (A., Hoyrup [2])

A compact pair (X, A) has strong computable type if for every oracle O and every copy (Y, B) of (X, A) in Q, if (Y, B) is semicomputable relative to O, then Y is computable relative to O. X has strong computable type if $(X, \emptyset)$ has it.

Examples

- The n-dimensional ball with its bounding sphere, Miller 2002 [6]

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The ϵ-surjection property

Definition (A., Hoyrup [1])

Let $\epsilon > 0$. A pair $(X, A) \subseteq Q$ satisfies the ϵ-surjection property if every continuous function $f : X \to X$ satisfying $d_X(f, \text{id}_X) < \epsilon$ and $f|_A = \text{id}_A$ is surjective.

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Example

The n-dimensional sphere.
The n-dimensional ball with its bounding sphere.

A necessary condition

Theorem (A., Hoyrup [2])

If $(X, A) \subseteq Q$ has strong computable type, then it satisfies the ϵ-surjection property for some $\epsilon > 0$.

Finite simplicial complexes

Theorem (A., Hoyrup [1])

For a finite simplicial pair (X, A) such that A has empty interior in X, the following statements are equivalent:

1. *(X, A) has (strong) computable type,*
2. *(X, A) satisfies the ϵ-surjection property for some $\epsilon > 0$.*

A question

For which other classes of spaces, satisfying the ϵ-surjection property is sufficient to have strong computable type?

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An answer: computably ANRs.

Absolute neighborhood retracts

Definition

Let (X, A) be a pair. A retraction $r : X \to A$ is a continuous function such that $r|_A = \mathrm{id}_A$. If a retraction exists, then we say that A is a retract of X.

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Definition

Let X be a Hausdorff compact space. X is an absolute neighborhood retract (ANR) if every copy X' of X in Q is retract of a neighborhood of X'.

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Computably absolute neighborhood retracts

Definition (A.)

A compact metrizable space X is a computably absolute neighborhood retract (computably ANR) if there exists a copy X_0 of X in the Hilbert cube and a computable retraction $r : U \to X_0$ for some open set U containing X_0.

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Fact (Collins [3])

The homology groups of computably ANRs are computable.

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- n-dimensional spheres are computably ANRs.
- More generally, finite simplicial complexes are computably ANRs.

An interesting question

Are compact manifolds with or without boundary computably ANRs?

A sufficient condition

Theorem (A.)

Let X be a computably ANR, if X satisfies the ϵ-surjection property then it has strong computable type.

Sketch of the proof: ingredients

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- Let V be an open set and $Y = X \setminus V$,
- Let (g_s) such that $g_s = f_j|_Y$ for some j and $g_s(Y) \subset U$,
- Take g_k such that $d_Y(h|_Y, g_k)$ small enough.

Sketch of the proof: an equivalence

1. $Y \neq X$,
2. There exists some l such that $r \circ g_l : Y \to X_0$ is non-surjective and $d_Y(r \circ g_l, g_k)$ is small.

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Finally, the proof works relative to any oracle.

Extension to pairs

Theorem (A.)

Let X be a computably ANR and $A \subset X$ be compact ANR, if (X, A) satisfies the ϵ-surjection property then it has strong computable type.

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Thank you!