A characterisation of functions computable in polynomial time and space over the reals with discrete ordinary differential equations

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Introduction

Introduction

- providing algebraic characterisations of polynomial complexity classes, without:
  - explicit bounds (Cobham)
  - several types of arguments (Bellantoni and Cook)
  - tiering (Leivant and Marion)...
- We are in the framework of implicit complexity

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- ... but here we are interested in functions over the real numbers

Our framework: Computable Analysis

- Computable real number
- Computable function over the reals
- Discrete derivation : $f : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$,

$$f'(x, y) = \frac{\partial f}{\partial x}(x, y) = f(x + 1, y) - f(x, y).$$

Previous work: characterising computable reals and sequences (MCU22)

Motivation behind Length-ODEs

Fig. 5. A continuous system before and after an exponential speed-up.

θ solution of:

$$y' = f(y)$$

with $f : \mathbb{R} \to \mathbb{R}$

ϕ solution of:

$$z = z'$$

$$y' = f(y)z$$

Motivation behind Length-ODEs

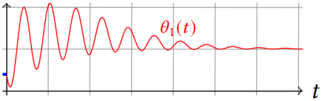

Fig. 5. A continuous system before and after an exponential speed-up.

Re-scaling: $\phi_1(t) = \theta(e^t)$

Motivation behind Length-ODEs

Now "time-complexity" is measured by the length of the solution curve of the ODE.

Invariance by rescaling

“Length ODE”?

Definition (Length ODE)

A function $\mathbf{f}$ is "length-ODE" definable (from u, g and h) if it is a solution of

$$f(0,\mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial \ell} = \mathbf{u}(\mathbf{f}(x,\mathbf{y}), \mathbf{h}(x,\mathbf{y}), x, \mathbf{y}). \qquad (1)$$

Formal synonym for right-hand side of (1):
$\mathbf{f}(x+1,\mathbf{y}) = \mathbf{f}(x,\mathbf{y}) + (\ell(x+1) - \ell(x)) \cdot \mathbf{u}(\mathbf{f}(x,\mathbf{y}), \mathbf{h}(x,\mathbf{y}), x, \mathbf{y})$

“Length ODE”?

A derivation with respect to the length corresponds to change of variable in the ODE. Variation when:

$$\ell(x+1) - \ell(x) \neq 0$$

Inspired by:

$$\frac{\delta f(x, \mathbf{y})}{\delta x} = \frac{\delta \ell(x)}{\delta x} \cdot \frac{\delta f(x, \mathbf{y})}{\delta \ell(x)}.$$

Example of length-ODE

$$
\begin{aligned}
f(0) &= 2 \\
\frac{\partial f}{\partial \ell}(x) &= f(x) \cdot f(x) - f(x)
\end{aligned}
$$

Example of length-ODE

$$f(0) = 2$$
$$\frac{\partial f}{\partial \ell}(x) = f(x) \cdot f(x) - f(x)$$

Unique solution : $f(x) = 2^{2^{\ell(x)}}$

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- We have $f(x) = F(z)$
- $F(z+1) = 2^{2^z + 2^z} = F(z)F(z)$
- Then, $F(z+1) = F(z) + (z+1-z)(F(z)F(z) - F(z))$

Effective Limit

Definition
$f : \mathbb{R} \to \mathbb{R}$ is an effective limit of $\bar{f} : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ if

$$\left| f(x) - \bar{f}(x, 2^n) \right| \leq 2^{-n}$$

Key observation : $\bar{f}$ computable in polynomial time $\Rightarrow f$ computable in polynomial time.

Algebraic characterisation of PTIME

We consider:

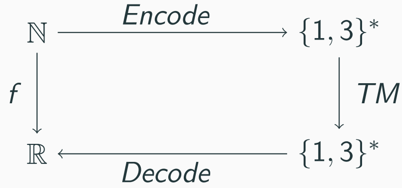
$$\overline{\mathbb{LDL}^{\bullet}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \times, \overline{\mathrm{cond}}(x), \frac{x}{2};$$

$$\textit{composition}, \textit{linear length ODE}, \textit{effective limit}]$$

Theorem (MCU22)

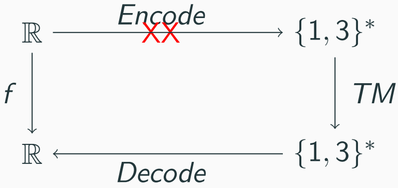
$$\mathrm{FPTIME} \cap \mathbb{R}^{\mathbb{N}} = \overline{\mathbb{LDL}^{\bullet}} \cap \mathbb{R}^{\mathbb{N}}$$

“Proof”



Characterising computable functions over the reals

What we cannot do



FPTIME for computable functions over the reals

We consider:

$$\overline{\mathbb{LDL}^{\circ}} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{linear length ODE}, \textit{effective limit}]$$

Theorem

$$\overline{\mathbb{LDL}^{\circ}} \cap \mathbb{R}^{\mathbb{R}} = \mathrm{FPTIME} \cap \mathbb{R}^{\mathbb{R}}$$

Toward FPSPACE: Robust linear ODE

Definition (Robust linear ODE)

A bounded function $\mathbf{f}$ is a robustly linear ODE definable (with $\mathbf{u}$ *essentially linear* in $\mathbf{f}(x, \mathbf{y})$, $\mathbf{g}$ and $\mathbf{h}$) if:

1. it is a solution of

$$\mathbf{f}(0, \mathbf{y}) = \mathbf{g}(\mathbf{y}) \quad \textit{and} \quad \frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} = \mathbf{u}(\mathbf{f}(x, \mathbf{y}), \mathbf{h}(x, \mathbf{y}), x, \mathbf{y}),$$

2. the ODE is (polynomially) numerically stable.

FPSPACE for computable functions over the reals

We consider:

$$\overline{\mathbb{RLD}^\circ} = [\mathbf{0}, \mathbf{1}, \pi_i^k, \ell(x), +, -, \tanh, \frac{x}{2}, \frac{x}{3};$$

$$\textit{composition}, \textit{robust linear ODE}, \textit{effective limit}]$$

Theorem

$$\overline{\mathbb{RLD}^\circ} \cap \mathbb{R}^\mathbb{R} = \mathsf{FPSPACE} \cap \mathbb{R}^\mathbb{R}$$

Conclusion

Conclusion

- We now have "natural" algebraic characterisations for functions over the reals computable in FPTIME and FPSPACE.
- Further works:
  - Going to continuous ODEs
  - Cover other complexity classes
  - See what we can say for more general functions