

On the complexity of learning programs

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- ▶ Given a prefix of a sequence of numbers

3, 9, 15, 21, ...,

one can ask how the sequence continues?

- ▶ Provided the input sequence is total computable, the answer could be a Gödel number for it.
- ▶ This and similar questions have been intensively studied in algorithmic learning theory.
- ▶ Gold proved 1967 that one cannot even learn the Gödel number in the limit, in the situation above.
- ▶ We want to classify the Weihrauch complexity of the above problem.
- ▶ In this way we get a better understanding of the mixture of topological and computability-theoretic features that are involved in this problem.



- ▶ Let $\varphi : \mathbb{N} \rightarrow \mathcal{P}$ be some standard Gödel numbering of the set \mathcal{P} of partial computable functions.

- ▶ We call the following problem the **Gödelization problem**

$$G : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, p \mapsto \{i \in \mathbb{N} : \varphi_i = p\},$$

where $\text{dom}(G)$ contains all total computable functions p .

- ▶ For our purposes the **Kolmogorov complexity** is the problem

$$K : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}, p \mapsto \min G(p),$$

with $\text{dom}(K) = \text{dom}(G)$.

- ▶ Hoyrup and Rojas (2017) have coined the following slogan:
The only useful additional information carried by a program compared to the natural number sequence it represents, is an upper bound on the Kolmogorov complexity of the sequence.

Variants of the Gödelization problem



- ▶ We also look at the following variant of G :

$$G_{\geq} : \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightrightarrows \mathbb{N}, (p, m) \mapsto \{i \in \mathbb{N} : \varphi_i = p\},$$

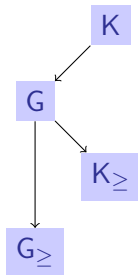
where $\text{dom}(G) = \{(p, m) : K(p) \leq m\}$.

- ▶ And we study the following variant of K :

$$K_{\geq} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, p \mapsto \{m \in \mathbb{N} : K(p) \leq m\},$$

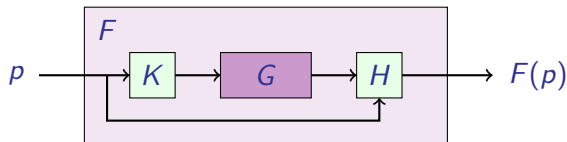
with $\text{dom}(K_{\geq}) = \text{dom}(G)$.

- ▶ These problems are related in the Weihrauch lattice as follows:



Weihrauch reducibility

Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq Z \rightrightarrows W$ be two multi-valued functions.



- ▶ f is **Weihrauch reducible** to g , $f \leq_W g$, if there are computable $H, K : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $H\langle \text{id}, GK \rangle \vdash f$ whenever $G \vdash g$.
- ▶ We write $f \leq_W^* g$ for the **continuous** version of Weihrauch reducibility, where H, K are chosen to be continuous.
- ▶ We write $f \leq_W^p g$ if H, K can be chosen to be **computable relative to** $p \in \mathbb{N}^{\mathbb{N}}$.
- ▶ \equiv_W , \equiv_W^* , and \equiv_W^p denote the corresponding equivalences.
- ▶ The distributive lattice induced by \leq_W is usually referred to as **Weihrauch lattice**.

Typical problems in the Weihrauch lattice



- ▶ **Limited principle of omniscience:**

$$\text{LPO} : \mathbb{N}^{\mathbb{N}} \rightarrow \{0, 1\}, \text{LPO}(p) = 1 : \iff p = \widehat{0}$$

- ▶ **Lesser limited principle of omniscience:**

$$\text{LLPO} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \{0, 1\}, \text{LLPO}\langle p_0, p_1 \rangle := \{i \in \{0, 1\} : p_i = \widehat{0}\},$$

with $\text{dom}(\text{LLPO}) = \{\langle p_0, p_1 \rangle \in \mathbb{N}^{\mathbb{N}} : \neg(p_0 \neq \widehat{0} \wedge p_1 \neq \widehat{0})\}$.

- ▶ **Closed choice on \mathbb{N} is**

$$\text{C}_{\mathbb{N}} : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}, p \mapsto \{n \in \mathbb{N} : (\forall k) p(k) \neq n\},$$

with $\text{dom}(\text{C}_{\mathbb{N}}) = \{p \in \mathbb{N}^{\mathbb{N}} : \text{range}(p) \subsetneq \mathbb{N}\}$,

- ▶ **Compact choice \mathbb{N} is**

$$\text{K}_{\mathbb{N}} : \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightrightarrows \mathbb{N}, (p, m) \mapsto \{n \leq m : (\forall k) p(k) \neq n\},$$

with $\text{dom}(\text{K}_{\mathbb{N}}) = \{(p, m) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} : \text{range}(p) \subsetneq \{0, \dots, m\}\}$.

- ▶ **Weak König's lemma:** $\text{WKL} : \subseteq \text{Tr} \rightrightarrows 2^{\mathbb{N}}, T \mapsto [T]$

- ▶ **Limit:** $\text{lim} : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \langle x_n \rangle \mapsto \lim_{n \rightarrow \infty} x_n$.

Borel complexity and Weihrauch complexity



The **jump** f' of a problem is a strengthening of f :

- ▶ a name of an input x for f' is a sequence (p_n) in $\mathbb{N}^{\mathbb{N}}$ that converge to a name $p \in \mathbb{N}^{\mathbb{N}}$ of an input in the sense of f .

Theorem (B. 2005, Pauly, de Brecht 2014 and Kihara 2015)

1. f is computably Σ_{n+2}^0 -measurable $\iff f \leq_W \text{lim}^{(n)}$.
2. f is computably $(\Sigma_{n+2}^0, \Sigma_{n+2}^0)$ -measurable $\iff f \leq_W C_{\mathbb{N}}^{(n)}$.

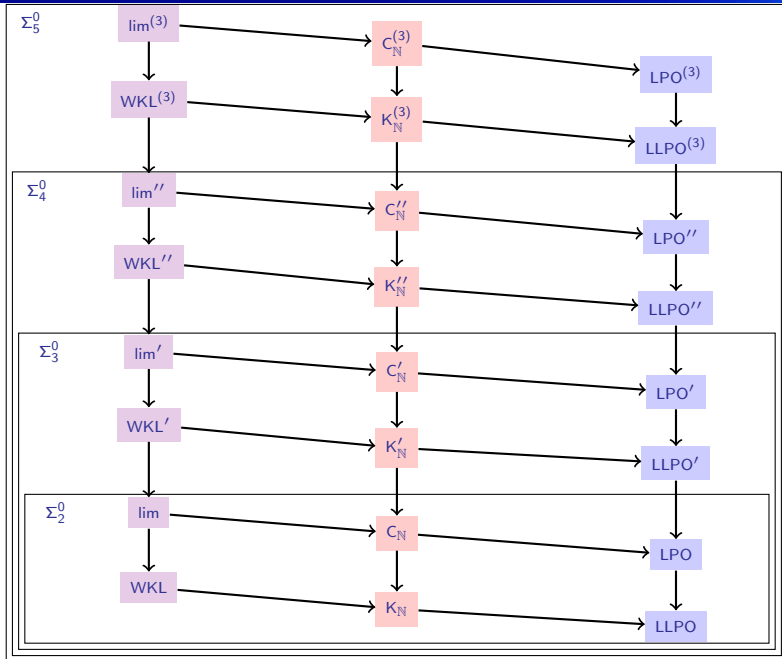
- ▶ Weihrauch complexity refines **Borel complexity** very much in the same way as many-one complexity refines arithmetical complexity.
- ▶ B. and Rakotoniaina (2017) have shown that

$$K_{\mathbb{N}} \leq_W C_{\mathbb{N}} \leq_W K'_{\mathbb{N}} \leq_W C'_{\mathbb{N}} \leq_W \dots$$

and concluded that this is the proper Weihrauch analogue of the **Paris-Harrington hierarchy** of **induction and boundedness problems**

$$B\Sigma_1^0 \leftarrow I\Sigma_1^0 \leftarrow B\Sigma_2^0 \leftarrow I\Sigma_2^0 \leftarrow \dots$$

Basic skeleton of Weihrauch complexity



Motivation for closed and compact choice as benchmarks

Recall that the **first-order part** of a problem f can be defined by

$${}^1f := \max_{\leq_W} \{g : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N} : g \leq_W f\}.$$

It was introduced by Dzhafarov, Solomon, and Yokoyama (2019).

Theorem (Valenti 2021, Soldà and Valenti 2023)

1. ${}^1(\text{lim}^{(n)}) \equiv_{sW} C_{\mathbb{N}}^{(n)}$, in particular ${}^1\text{lim} \equiv_{sW} C_{\mathbb{N}}$,
2. ${}^1(\text{WKL}^{(n)}) \equiv_{sW} K_{\mathbb{N}}^{(n)}$, in particular ${}^1\text{WKL} \equiv_{sW} K_{\mathbb{N}}$.

By a result of Westrick (2021) the **diamond** can be characterized by

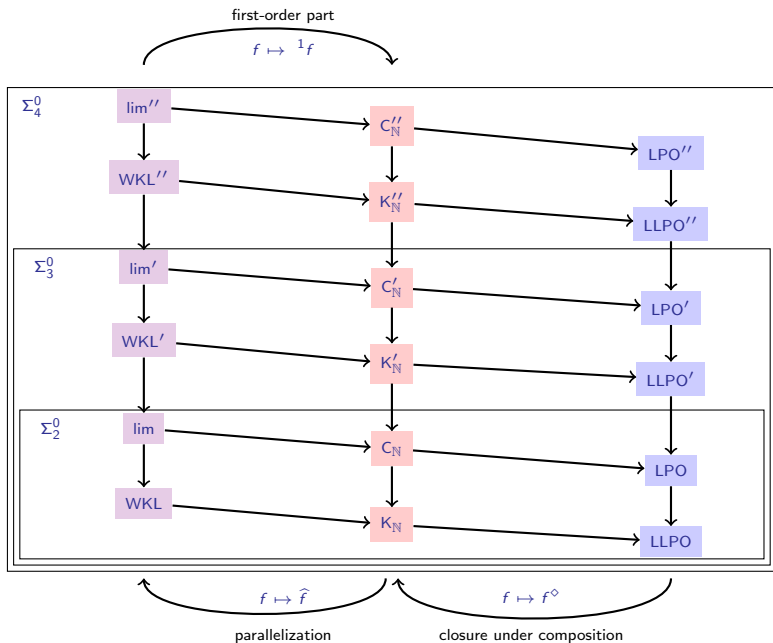
$$f^{\diamond} := \max_{\leq_W} \{g : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}} : f \leq_W g \star g \leq_W g\}.$$

It was introduced by Neumann and Pauly (2018).

Proposition

- ▶ $\text{LPO}^{\diamond} \equiv_W C_{\mathbb{N}}$ *(Neumann and Pauly 2018)*
- ▶ $\text{LLPO}^{\diamond} \equiv_W K_{\mathbb{N}}$ *(Soldà and Valenti 2023)*

Basic skeleton of Weihrauch complexity





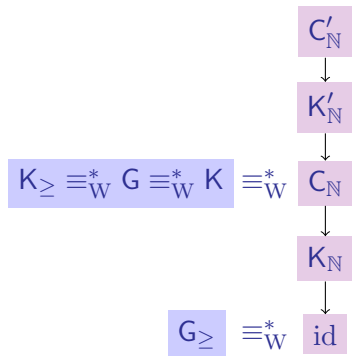
Weihrauch degree	Reverse mathematics axioms
$C_{\mathbb{N}^{\mathbb{N}}}$	ATR_0
lim^{\diamond}	ACA_0
WKL	WKL_0^*
$C_{\mathbb{N}}^{(n)}$	$I\Sigma_{n+1}^0$
$K_{\mathbb{N}}^{(n)}$	$B\Sigma_{n+1}^0$
id	RCA_0^*

Theorem (B., de Brecht and Pauly 2012)

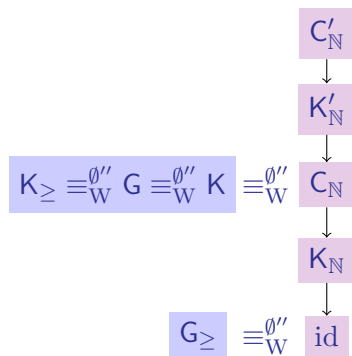
1. f is limit computable $\iff f \leq_W \text{lim}$.
2. f is finite mind change computable $\iff f \leq_W C_{\mathbb{N}}$.
3. f is non-deterministically computable $\iff f \leq_W \text{WKL}$.

- ▶ Gold's result can be translated into $G \not\leq_W C_{\mathbb{N}}$.
- ▶ We will use the problems $K_{\mathbb{N}}$ and $C_{\mathbb{N}}$ as a benchmark to classify the Gödel problem.

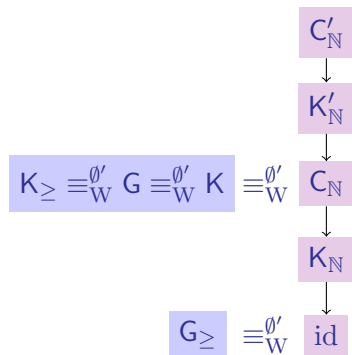
The topological situation



- ▶ The equivalence $K_{\geq} \equiv_W^* G$ validates Hoyrup and Rojas slogan topologically.
- ▶ Which is the minimal oracle among $\emptyset, \emptyset', \emptyset'', \dots$ that validates the picture above in place of $*$?



- ▶ The oracle \emptyset'' makes totality decidable and this yields easy proofs of the equivalences.
- ▶ Surprisingly, this can also be done with \emptyset' , but the proofs are slightly more difficult in this case.



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Proposition

$$K \leq_{\mathcal{W}}^{\emptyset'} C_{\mathbb{N}}.$$

Proof.

- ▶ We go through all Gödel numbers $i = 0, 1, 2, \dots$ one by one.
- ▶ For each i we check for each $n = 0, 1, 2, \dots$ whether $n \in \text{dom}(\varphi_i)$ (with the help of the halting problem) and whether $\varphi_i(n) = p(n)$.
- ▶ If so, then we write i to the output q and we move on to the next n .
- ▶ If one of these tests fails, then we move on to the next i .
- ▶ This procedure stops going to the next i when the smallest i with $\varphi_i = p$ is reached.
- ▶ Altogether, this gives a finite mind change computation for K .



Proposition

$$C_{\mathbb{N}} \leq_W^{\emptyset'} K_{\geq}.$$

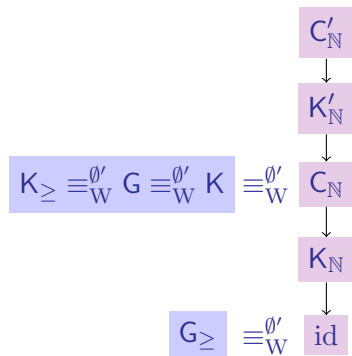
Proof.

- ▶ We use a variant of the set of **random natural numbers**:

$$R := \{\langle k, n \rangle \in \mathbb{N} : \min\{i \in \mathbb{N} : \varphi_i(k) = n\} \geq n\}.$$

- ▶ For each k there are infinitely many n with $\langle k, n \rangle \in R$.
- ▶ R is co-c.e. and hence $R \leq_T \emptyset'$.
- ▶ We use the **boundedness problem** $B \equiv_W C_{\mathbb{N}}$, which is the problem: given a monotone increasing bounded sequence $p \in \mathbb{N}^{\mathbb{N}}$, find an upper bound $b \in \mathbb{N}$.
- ▶ We prove $B \leq_W^R K_{\geq}$: inspecting the numbers $p(0), p(1), p(2), \dots$ we construct $q(0), q(1), q(2), \dots$ such that $b = K(q)$ is an upper bound for p .
- ▶ This can be done such that q is eventually constant and hence actually computable.





- ▶ We have established the upper equivalences.
- ▶ We still need to prove G_{\ge} is computable relative to the halting problem.



Proposition

G_{\geq} is computable with respect to the halting problem \emptyset' .

Proof. We use a variant of the **amalgamation technique**.

- ▶ We consider the **compatibility relation** on \mathcal{P} :

$$f \approx g : \iff (\forall n \in \text{dom}(f) \cap \text{dom}(g)) f(n) = g(n).$$

- ▶ $C := \{\langle i, j \rangle \in \mathbb{N} : \varphi_i \approx \varphi_j\}$ is co-c.e. and hence $C \leq_T \emptyset'$.

- ▶ Let (p, m) be an input for G_{\geq} , i.e., $K(p) \leq m$.

- ▶ For $i \leq m$ that we consider the **pockets**:

$$P_i := \{j \leq m : \varphi_i \approx \varphi_j\}$$

- ▶ P_i is called **compatible**, if $\varphi_{j_0} \approx \varphi_{j_1}$ holds for all $j_0, j_1 \in P_i$.

- ▶ Among P_0, \dots, P_m we remove all incompatible pockets and all double occurrences of the same pocket.

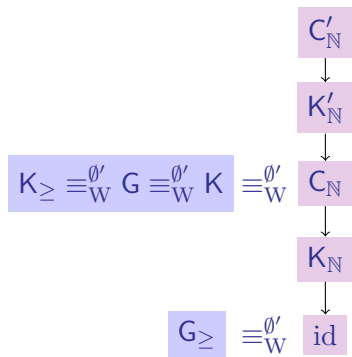
- ▶ This yields a list of P_{i_0}, \dots, P_{i_k} of pairwise different pockets, which are all compatible by themselves.

Computability with respect to the halting problem



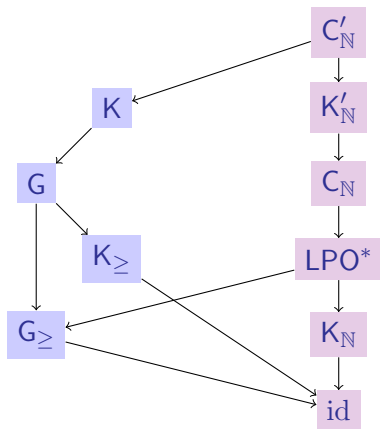
- ▶ No pocket in our list is a subset of another pocket.
- ▶ Among the pockets P_{i_0}, \dots, P_{i_k} in our list
 1. exactly one contains at least one code j with $\varphi_j = p$ and all codes j in this pocket satisfy $\varphi_j \approx p$,
 2. all other pockets contain at least one j with $\varphi_j \not\approx p$.
- ▶ P_i is called **compatible with** p , if $p \approx \varphi_j$ for all $j \in P_i$.
- ▶ 1. and 2. guarantee that there is exactly one pocket P_i among the P_{i_0}, \dots, P_{i_k} that is compatible with p and contains a Gödel number of p .
- ▶ A prefix of p is sufficient to identify P_i as we just need to find an incompatible member in all the other pockets.
- ▶ From the index i we can compute a Gödel number $r(i)$ of p : for each input $n \in \mathbb{N}$ we search for some $j \in P_i$ such that $n \in \text{dom}(\varphi_j)$ and we produce $\varphi_j(n)$ as result.
- ▶ Hence, $r(i) \in G_{\geq} \langle p, m \rangle$. (We note that $r(i) \leq m$ is not required and might not hold.)





- ▶ We now want to study the situation in the computable case.
- ▶ We know $G \not\leq_W C_N$ by Gold (1967) and $G_{\ge} \leq_W C_N$ by Freivald and Wiehagen (1979).

The computability-theoretic situation



The computability-theoretic situation



- ▶ $K \leq_W C'_N$ can be proved observing that $C'_N \equiv_W \liminf_N$. We just write all Gödel numbers i onto the output that match the input for longer and longer prefixes of the input p . The least cluster point is the smallest Gödel number of p .
- ▶ $K_{\geq} \not\leq_W K'_N$ can be proved by a finite extension construction using that $K'_N \equiv_W \text{BWT}_N$ (the Bolzano-Weierstraß theorem on \mathbb{N}).
- ▶ Hence the classification of $K_{\geq} \leq_W G \leq_W K$ is optimal with respect to our benchmark problems.
- ▶ $G_{\geq} \leq_W \text{LPO}^*$ can be proved with the amalgamation technique.
- ▶ $G_{\geq} \not\leq_W K_N$ can be proved with a finite extension construction.
- ▶ G_{\geq} is hence continuous, but not computable.
- ▶ The problems G_{\geq}, K_{\geq}, G and K can all be separated from each other with respect to \leq_W .



- ▶ By \widehat{G} we denote the **parallelization** of G
- ▶ By $G \star G$ we denote the **compositional product** of G by itself
- ▶ By G^* we denote the **finite parallelization** of G
- ▶ By $f|_c$ we denote the **restriction to computable inputs** of f

- ▶ $\widehat{G}|_c \equiv_W G <_W \widehat{G}$ (parallelization)
- ▶ $(G \star G)|_c \equiv_W G$ (compositional products)
- ▶ $G^* \equiv_W G$ (finite parallelization)

Question

Does $G \star G \equiv_W G$ hold?

Proposition

$DIS \not\leq_W G$, but $LPO \leq_W K$.

Proof. $DIS \leq_W G$ would imply $NON \leq_W \widehat{G}$, since $\widehat{DIS} \equiv_W NON$. But since $\widehat{G}|_c \leq_W G$, this is impossible!

$LPO \leq_W K$ is easy to see, as there is a specific smallest Gödel number i of the zero sequence $p \in \mathbb{N}^{\mathbb{N}}$. □

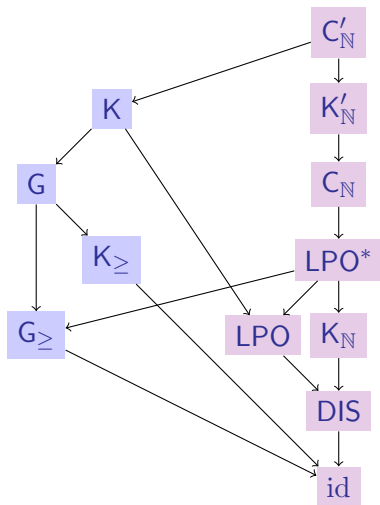
DIS is the weakest natural discontinuous problem with respect to topological Weihrauch reducibility (in $ZF+DC+AD$). Hence, Gödelization G has no useful natural lower bounds (besides id)!

Corollary

G is effectively discontinuous, but not computably so.

This means $DIS \leq_W^* G$, but $DIS \not\leq_W G$.

The computability-theoretic situation





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