Non-compact manifolds and computable type

Matea Čelar
University of Zagreb

CCA 2023, Dubrovnik, September 2023

COMPUTABLE METRIC SPACE

A **computable metric space** is a triple (X, d, α), where (X, d) is a metric space and $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ is a sequence in X whose image is dense in (X, d) such that the function $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

Let $(B_i)_{i \in \mathbb{N}}$ be a fixed enumeration of open balls in (X, d) with rational radii centered at α_i. By $\hat{B}_i$ we denote the closed ball corresponding to B_i.

Let $(\Delta_j)_{j \in \mathbb{N}}$ be some fixed effective enumeration of all finite subsets of $\mathbb{N}$. By J_j we denote the set

$$\bigcup_{i \in \Delta_j} B_i.$$

COMPUTABILITY OF SETS

Let $(X, d, (\alpha_i)_{i\in\mathbb{N}})$ be a computable metric space and $S \subseteq X$. We say that S is:

- **computable compact** if S is compact and $S = \emptyset$ or it can be effectively approximated by finite sets of special points α_i
- **computably enumerable (closed)**, if S is closed and the set $\{i \in \mathbb{N} \mid S \cap B_i \neq \emptyset\}$ is c.e.
- **semicomputable compact** if S is compact and the set $\{i \in \mathbb{N} \mid S \subseteq J_i\}$ is c.e.

It holds that

$$\boxed{S \text{ computable compact} \iff S \text{ c.e. and semicomputable compact}} \quad (1)$$

We say that $S \subseteq X$ is **semicomputable** if

(i) $S \cap B$ is compact for each closed ball B in (X, d)

(ii) $\{(i, j) \in \mathbb{N}^2 \mid S \cap \hat{B}_i \subseteq J_j\}$ is c.e.

It holds that:

$$S \text{ compact and semicomputable} \quad \Longleftrightarrow \quad S \text{ semicomputable compact} \tag{2}$$

(1) and (2) motivate the following definition:

$$S \textbf{ computable} \quad :\Longleftrightarrow \quad S \text{ semicomputable and c.e.} \tag{3}$$

COMPUTABLE TYPE

Topological pair (A, B) has **computable type** if, for any copy (S, T) of (A, B) in a computable metric space, if S and T are semicomputable compact, then S is computable compact.

Topological space A has computable type if $(A, \emptyset)$ has computable type.

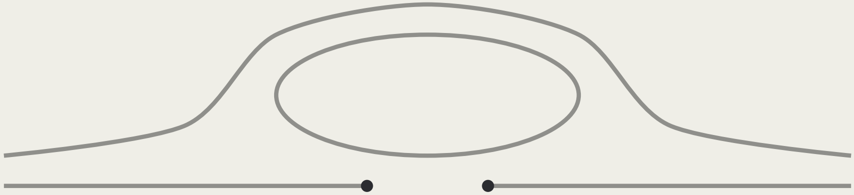
Definition

Topological pair (A, B) has **computable type** if, for any copy (S, T) of (A, B) in a computable metric space, if S and T are semicomputable, then S is computable.

SOME KNOWN RESULTS

Theorem 1 (Burnik and Iljazović 2014)

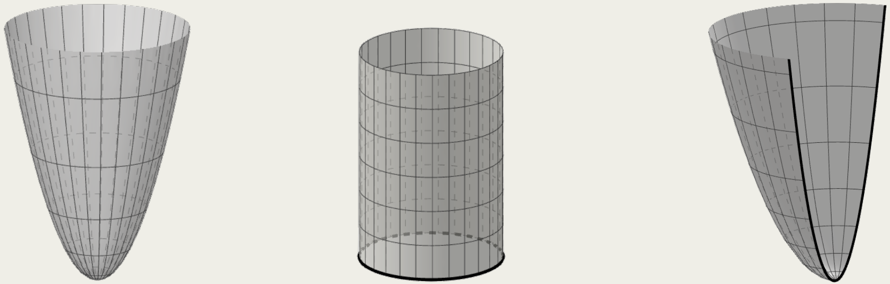
Let (X, d, α) be a computable metric space. Let M be a 1-manifold with boundary in this space such that M has finitely many components. Suppose M and ∂M are semicomputable. Then M is computable.



SOME KNOWN RESULTS

Theorem 2 (Iljazović and Sušić 2018)

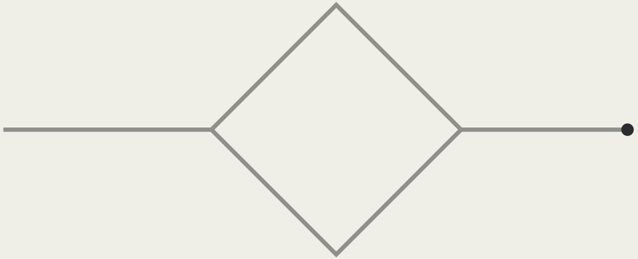
Let (X, d, α) be a computable metric space and let K be a manifold with boundary in this space. Suppose K and ∂K are semicomputable. Then K is computable if there exists an open set U in K such that $\overline{U}$ is compact in K and $K \setminus U$ is homeomorphic to $\mathbb{R}^n \setminus B(0,1)$ or $\mathbb{H}^n$.



SOME KNOWN RESULTS

Theorem 3 (Iljazović 2020)

Let (X, d, α) be a computable metric space and let S be a semicomputable set in this space. Suppose S, as a subspace of (X, d), is a generalized graph such that the set E of all endpoints of S is semicomputable in (X, d, α). Then S is computable in (X, d, α).



COMPACTIFICATION

Suppose $(X, \mathcal{T})$ is a topological space.

A subset U of X is a **neighborhood of infinity** if $\overline{X \setminus U}$ is compact.

One-point compactification of $(X, \mathcal{T})$ is a topological space $X^\infty := (Y, \mathcal{S})$ where $Y = X \cup \{\infty\}$, $\infty \notin X$ and

$$\mathcal{S} = \mathcal{T} \cup \{\{\infty\} \cup U \mid U \in \mathcal{T} \text{ and } X \setminus U \text{ is compact in } (X, \mathcal{T})\}.$$

PSEUDOCOMPACTIFICATION

Suppose (X, d, α) is a computable metric space and $Y = X \cup \{\infty\}$, where $\infty \notin X$. Let

$$\mathcal{S} = \mathcal{T}_d \cup \{\{\infty\} \cup U \mid U \text{ is open in } (X, d) \text{ and } X \setminus U \text{ is bounded in } (X, d)\}.$$

For $i \in \mathbb{N}$ let

$$I_i = \begin{cases} B_{\frac{i}{2}}, & i \in 2\mathbb{N} \\ \{\infty\} \cup (X \setminus \hat{B}_{\frac{i-1}{2}}), & i \in 2\mathbb{N}+1 \end{cases}$$

We call the computable topological space $(Y, S, (I_i)_{i \in \mathbb{N}})$ a **pseudocompactification** of the computable metric space (X, d, α).

Theorem 4 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (I_i))$ be its pseudocompactification. Let K be a semicomputable set in (X, d, α). Suppose the metric space (X, d) is unbounded.

(i) If K is compact in (X, d), then K is semicomputable in $(Y, \mathcal{S}, (I_i))$.

(ii) If K is not compact in (X, d), then $K \cup \{\infty\}$ is semicomputable in $(Y, \mathcal{S}, (I_i))$.

Theorem 5 (Iljazović and Sušić 2018)

Let (X, d, α) be a computable metric space and let $(Y, \mathcal{S}, (I_i))$ be its pseudocompactification. Suppose $K \subseteq X$ is such that $K \cup \{\infty\}$ is a c.e. set in $(Y, \mathcal{S}, (I_i))$. Then K is c.e. in (X, d, α).

LOCAL COMPUTABILITY

Suppose $T \subseteq S$. We say that T is **computably enumerable up to** S if there exists a r.e. set $\Omega \subseteq \mathbb{N}$ such that

$$\{i \in \mathbb{N} \mid B_i \cap T \neq \emptyset\} \subseteq \Omega \subseteq \{i \in \mathbb{N} \mid B_i \cap S \neq \emptyset\}.$$

We say that S is **computably enumerable at** $x \in S$ if x has a neighborhood in S which is c.e. up to S.

Topological space A has **local computable type for compact spaces** if a semicomputable compact set is c.e. at any point which has a neighborhood homeomorphic to A.

LOCAL COMPUTABILITY

Definition

Topological space A has **local computable type** if a semicomputable set is c.e. at any point which has a neighborhood homeomorphic to A.

Lemma 6

If a topological space has local computable type for compact spaces, then it has local computable type.

LOCAL COMPUTABILITY AT INFINITY

Definition

We say that S is **computably enumerable at infinity** if S contains a neighborhood of infinity which is c.e. up to S.

Lemma 7

Suppose S is a semicomputable set in a computable metric space. If S is computably enumerable at infinity and at every $x \in S$, then S is computable.

COLLAR NEIGHBORHOODS

Proposition 8

Suppose S is a semicomputable set in a computable metric space which contains a closed neighborhood of infinity homeomorphic to $Q \times [0, 1\rangle$ for some compact space Q. If the cone $\mathsf{C}(Q)$ has local computable type, then S is computable at infinity.

CYLINDERS

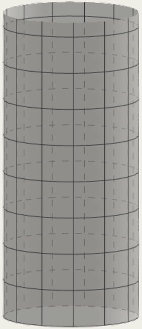
Corollary 9

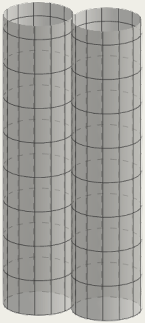
Suppose Q is a compact manifold such that the cone $\mathsf{C}(Q)$ has local computable type. Then $Q \times \mathbb{R}$ has computable type.

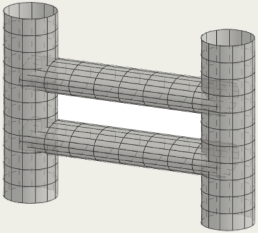
Corollary 10

$\mathbb{S}^n \times \mathbb{R}$ has computable type.

MORE EXAMPLES







REFERENCES

- Burnik, K. and Z. Iljazović (2014). "Computability of 1-manifolds". In: *Logical Methods in Computer Science* 10.2:8, pp. 1–28.
- Guilbault, Craig R. (2016). "Ends, Shapes, and Boundaries in Manifold Topology and Geometric Group Theory". In: *Topology and Geometric Group Theory*. Ed. by Michael W. Davis et al. Cham: Springer International Publishing, pp. 45–125. ISBN: 978-3-319-43674-6.
- Iljazović, Z. (2020). "Computability of graphs". In: *Mathematical Logic Quarterly* 66, pp. 51–64.
- Iljazović, Z. and I. Sušić (2018). "Semicomputable manifolds in computable topological spaces". In: *Journal of Complexity* 45, pp. 83–114.

REFERENCES

Iljazović, Zvonko and Takayuki Kihara (2021). "Computability of Subsets of Metric Spaces". In: *Handbook of Computability and Complexity in Analysis*. Ed. by Vasco Brattka and Peter Hertling. Cham: Springer International Publishing, pp. 29–69. ISBN: 978-3-030-59234-9. DOI: 10.1007/978-3-030-59234-9_2. URL: https://doi.org/10.1007/978-3-030-59234-9_2.

Miller, J.S. (2002). "Effectiveness for Embedded Spheres and Balls". In: *Electronic Notes in Theoretical Computer Science* 66, pp. 127–138.

Rotman, J. (1998). *An Introduction to Algebraic Topology*. Graduate Texts in Mathematics. Springer New York. ISBN: 9780387966786. URL: https://doi.org/10.1007/978-1-4612-4576-6.

Thank you!