Reordered Computable Numbers

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Conventions in this talk:

- A "sequence" is a sequence of real numbers.
- A "computable sequence" is a computable sequence of rational numbers.

Definition

Let $f : \mathbb{N} \to \mathbb{N}$ be a function which tends to infinity. We define the function $u_f : \mathbb{N} \to \mathbb{N}$ by:

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- There exists a unique non-decreasing and unbounded function $f^*\colon \mathbb{N} \to \mathbb{N}$ with $u_{f^*} = u_f$.
- In general, both u_f and f^* are not computable even if f is computable.

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1. A function $s\colon \mathbb{N} \to \mathbb{N}$ is called a *modulus of convergence* of $(x_n)_n$ if for all $n \in \mathbb{N}$ and for all $i \geq s(n)$ we have $|x - x_i| < 2^{-n}$.

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2. We say that the sequence $(x_n)_n$ *converges computably* to x if it has a computable modulus of convergence.

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Every computable number is left-computable, but the converse is not true.

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- x_A is left-computable if A computably enumerable, but the converse is not true.

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Definition

A real number $x \in [0, 2]$ is called *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x_A = x$.

For the remainder of this talk we will only consider positive real numbers.

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It is easy to see that for every left-computable number there exists a computable function $g\colon \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{k=0}^{\infty} 2^{-g(k)}$ converges to it.

Proposition

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1. *For every bijective function* $\sigma : \mathbb{N} \to \mathbb{N}$*, the rearranged series* $\sum_{k=0}^{\infty} 2^{-f(\sigma(k))}$ *also converges, toward the same limit.*
2. *If* $s : \mathbb{N} \to \mathbb{N}$ *is a modulus of convergence, then* s *is also a modulus of convergence for the reordered series* $\sum_{k=0}^{\infty} 2^{-f^*(k)}$.

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2. *If* $s : \mathbb{N} \to \mathbb{N}$ *is a modulus of convergence, then* s *is also a modulus of convergence for the reordered series* $\sum_{k=0}^{\infty} 2^{-f^*(k)}$*.*
3. *The following are equivalent:*
   - *The series* $\sum_{k=0}^{\infty} 2^{-f^*(k)}$ *converges computably.*
   - *The series* $\sum_{k=0}^{\infty} u_f(k) \cdot 2^{-k}$ *converges computably.*

Definition

We say that a real number x is *reordered computable* if there exists a computable function $f \colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$ such that the reordered series $\sum_{k=0}^{\infty} 2^{-f^*(k)}$ converges computably.

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- By definition, every reordered computable number is a left-computable number.
- For example, every strongly left-computable number is reordered computable.

Proposition

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Theorem

Let x and y be left-computable numbers with $x \leq_S y$. If y is reordered computable, then x is reordered computable as well.

Theorem

Let x be a left-computable number such that there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$ and $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} < 2$. Then x is reordered computable.

Lemma

Let $(a_n)_n$ be a sequence with $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then the series $\sum_{k=0}^{\infty} a_k$ converges absolutely, and there exists a rational number $q < 1$ with $\sum_{k=n+1}^{\infty} |a_k| \leq \frac{q^{n+1}}{1-q}$ for all $n \in \mathbb{N}$.

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Let $(a_n)_n$ *be a sequence with* $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. *Then the series* $\sum_{k=0}^{\infty} a_k$ *converges absolutely, and there exists a rational number* $q < 1$ *with* $\sum_{k=n+1}^{\infty} |a_k| \leq \frac{q^{n+1}}{1-q}$ *for all* $n \in \mathbb{N}$.

Lemma

Let $h : \mathbb{N} \to \mathbb{N}$ *be a function with* $\limsup_{n\to\infty} \sqrt[n]{h(n)} < 2$. *Then the series* $\sum_{k=0}^{\infty} h(k) \cdot 2^{-k}$ *converges computably.*

Proof.

By assumption, we have $\limsup_{n\to\infty} \sqrt[n]{h(n)} < 2$. This also implies $\limsup_{n\to\infty} \sqrt[n]{h(n)\cdot 2^{-n}} < 1$. Hence, the series $\sum_{k=0}^{\infty} h(k)\cdot 2^{-k}$ converges, and there is a rational number $q < 1$ with $\sum_{k=n+1}^{\infty} h(k)\cdot 2^{-k} \leq \frac{q^{n+1}}{1-q}$ for all $n \in \mathbb{N}$. Define the function $s\colon \mathbb{N} \to \mathbb{N}$ by $s(n) := \min\left\{m \in \mathbb{N} \mid \frac{q^{m+1}}{1-q} < 2^{-n}\right\}$ for all $n \in \mathbb{N}$. Clearly, s is computable, and we claim that s is a modulus of convergence for the series. Considering an arbitrary $n \in \mathbb{N}$, we obtain:

$$\sum_{k=s(n)+1}^{\infty} h(k)\cdot 2^{-k} \leq \frac{q^{s(n)+1}}{1-q} < 2^{-n}$$

□

Corollary

Let $f : \mathbb{N} \to \mathbb{N}$ *be a computable function which tends to infinity with* $\limsup_{n\to\infty} \sqrt[n]{u_f(n)} < 2$. *Then the series* $\sum_{k=0}^{\infty} 2^{-f(k)}$ *converges and its limit is a reordered computable number.*

Theorem

Let $A \subseteq \mathbb{N}$ be an infinite set such that the number x_A is left-computable. If A is not immune, then x_A is reordered computable.

Definition (Hertling, Janicki (2023))

1. A sequence $(x_n)_n$ is called *nearly computably convergent* if it converges and for every computable increasing function $s : \mathbb{N} \to \mathbb{N}$ the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges computably.

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2. A real number is called *nearly computable* if there exists a computable sequence converging nearly computably to it.

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2. A real number is called *nearly computable* if there exists a computable sequence converging nearly computably to it.

Every computable number is nearly computable, and it follows from a theorem of Downey and LaForte (2002) that there exists a left-computable number which nearly computable but not computable.

Proposition (Hertling, Janicki (2023))

For a left-computable number x the following are equivalent:

1. *x is nearly computable.*
2. *For every computable increasing sequence $(x_n)_n$ converging to x, the sequence $(x_{n+1} - x_n)_n$ converges computably.*

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1. *x is nearly computable.*
2. *For every computable increasing sequence $(x_n)_n$ converging to x, the sequence $(x_{n+1} - x_n)_n$ converges computably.*

Theorem

Let x be a reordered computable number which is nearly computable. Then x is even computable.

Proof.

Since x is reordered computable, there exists a computable function $f\colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=0}^{\infty} 2^{-f(k)} = x$ and a computable function $r\colon \mathbb{N} \to \mathbb{N}$ with $\sum_{k=r(n)+1}^{\infty} u_f(k) \cdot 2^{-k} < 2^{-n}$ for all $n \in \mathbb{N}$. Since x is nearly computable, there exists a computable function $s\colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all $i \geq s(n)$ we have $f(i) > n$. Define the function $t\colon \mathbb{N} \to \mathbb{N}$ by $t(n) := s(r(n))$. Obviously, t is computable, and we claim that even the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges computably. Considering some arbitrary $n \in \mathbb{N}$, we obtain:

$$\sum_{k=t(n)+1}^{\infty} 2^{-f(k)} = \sum_{k=s(r(n))+1}^{\infty} 2^{-f(k)} \leq \sum_{k=r(n)+1}^{\infty} u_f(k) \cdot 2^{-k} < 2^{-n}$$

Hence, the series $\sum_{k=0}^{\infty} 2^{-f(k)}$ converges computably and x is a computable number. □

Corollary

Every reordered computable number is not Martin-Löf random.

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Proof.

Let Ω be a left-computable number which is Martin-Löf random. It is well-known that for every left-computable number x we have $x \leq_S \Omega$. Since the reordered computable numbers are closed downwards under the Solovay reduction and not every left-computable number is reordered computable, Ω is not reordered computable as well. □

Definition (Hertling, Hölzl, Janicki (2023))

A real number x is called *regainingly approximable* if there exists a computable increasing sequence $(x_n)_n$ converging to x with $x - x_n < 2^{-n}$ for infinitely many $n \in \mathbb{N}$.

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Obviously, every regainingly approximable is left-computable and every computable number is regainingly approximable, but none of these implications can be reversed (Hertling, Hölzl, Janicki (2023)).

Theorem

Let x be a reordered computable number. Then there exist regainingly approximable numbers α and β with $\alpha + \beta = x$.

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Let x be a reordered computable number. Then there exist regainingly approximable numbers α and β with $\alpha + \beta = x$.

This result is a strengthening of the splitting theorem by Hertling, Hölzl and Janicki (2023).

Thanks for your attention!