Benign approximations and non-speedability

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Conventions in this talk:

- A "sequence" is a sequence of real numbers.
- A "computable sequence" is a computable sequence of rational numbers.

Definition

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1. A function $f : \mathbb{N} \to \mathbb{N}$ is called a *modulus of convergence* of $(x_n)_n$ if for all $n \in \mathbb{N}$ and for all $m \geq f(n)$ we have $|x - x_m| < 2^{-n}$.

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2. We say that the sequence $(x_n)_n$ *converges computably* to x if it has a computable modulus of convergence.

Proposition

For a real number x the following are equivalent:

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A real number satisfying these properties is called *left-computable*.

Definition (Hertling, Hölzl, Janicki (2023))

A real number x is called *regainingly approximable* if there exists a computable non-decreasing sequence $(x_n)_n$ converging to x with $x - x_n < 2^{-n}$ for infinitely many $n \in \mathbb{N}$.

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A real number x is called *regainingly approximable* if there exists a computable non-decreasing sequence $(x_n)_n$ converging to x with $x - x_n < 2^{-n}$ for infinitely many $n \in \mathbb{N}$.

By definition, every regainingly approximable number is left-computable, but the converse is not true (Hertling, Hölzl, Janicki (2023)).

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Every computable number is regainingly approximable, but the converse is not true (Hertling, Hölzl, Janicki (2023)).

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1. A sequence $(x_n)_n$ is called *nearly computably convergent* if it converges and for every computable increasing function $s : \mathbb{N} \to \mathbb{N}$ the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges computably.

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2. A real number is called *nearly computable* if there exists a computable sequence converging nearly computably to it.

Every computable number is nearly computable, and it follows from a theorem of Downey and LaForte (2002) that there exists a left-computable number which nearly computable but not computable.

In this talk we are interested

- only in left-computable numbers and
- in the relationship between nearly computable numbers and regainingly approximable numbers.

Theorem (Main Result 1)

There exists a regainingly approximable number which is nearly computable, but not computable.

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This example is built using an infinite injury construction.

Theorem (Main Result 2)

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There exists a left-computable number which is nearly computable but not regainingly approximable.

This example is also built using an infinite injury construction.

Finally, there are also regainingly approximable numbers which are not nearly computable.

Definition

For a set $A \subseteq \mathbb{N}$ we define a real number in the interval $[0, 1]$ via:

$$x_A := \sum_{n \in A} 2^{-(n+1)}$$

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- x_A is computable if and only if A is computable.
- x_A is left-computable if A computably enumerable, but the converse is not true.

Definition

A real number $x \in [0, 1]$ is called *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x_A = x$.

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It follows from a theorem of Downey, Hirschfeldt and LaForte (2001) that every strongly left-computable number that is nearly computable is computable.

Theorem (Hertling, Hölzl, Janicki (2023))

There exists a regainingly approximable number which is strongly left-computable, but not computable.

Theorem (Hertling, Hölzl, Janicki (2023))

There exists a regainingly approximable number which is strongly left-computable, but not computable.

Corollary

There exists a regainingly approximable number which not nearly computable.

It turns out that the second main result is a key to solve an open question stated by Merkle and Titov (2020).

Definition (Merkle, Titov (2020))

Let x be a left-computable number. x is called *speedable* if there exists a constant $\rho \in \,]0,1[$ and a computable increasing sequence $(x_n)_n$ converging to x such that there are infinitely many $n \in \mathbb{N}$ with $\frac{x - x_{n+1}}{x - x_n} \leq \rho$.

Theorem (Barmpalias, Lewis-Pye (2017); Merkle, Titov (2020))

Every left-computable number which is speedable is not Martin-Löf random.

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Every left-computable number which is speedable is not Martin-Löf random.

Open Question: Is the converse true as well?

Proposition

For every left-computable number x the following are equivalent:

1. *x is speedable.*
2. *There exists a constant $\rho \in \,]0,1[$ and a computable increasing sequence of rational numbers $(x_n)_n$ converging to x such that there are infinitely many $n \in \mathbb{N}$ with $\frac{x_{n+1}-x_n}{x-x_n} \geq \rho$.*

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Proof.

Suppose that x is speedable, that is, by definition, there exists a constant $\rho' \in \,]0,1[$ and a computable increasing sequence $(x_n)_n$ converging to x with $\frac{x-x_{n+1}}{x-x_n} \leq \rho'$ for infinitely many $n \in \mathbb{N}$. If we let $\rho := 1 - \rho'$, this is equivalent to $\frac{x_{n+1}-x_n}{x-x_n} = 1 - \frac{x-x_{n+1}}{x-x_n} \geq \rho$ for infinitely many $n \in \mathbb{N}$. □

In other words, a left-computable number x is non-speedable if and only if for every computable increasing sequence $(x_n)_n$ converging to x the sequence $\left(\frac{x_{n+1}-x_n}{x-x_n}\right)_n$ converges to zero.

Proposition

Every regainingly approximable number is speedable.

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Proof.

Let x be regainingly approximable. Then, by definition, there exists a computable non-decreasing sequence $(x_n)_n$ converging to x with $x - x_n < 2^{-n}$ for infinitely many $n \in \mathbb{N}$. Define the sequence $(y_n)_n$ by $y_n := x_n - 2^{-n}$ for all $n \in \mathbb{N}$. This sequence is computable, increasing and also converges to x. Then, for every $n \in \mathbb{N}$ with $x - x_n < 2^{-n}$, we have

$$\frac{y_{n+1} - y_n}{x - y_n} = \frac{(x_{n+1} - x_n) + 2^{-(n+1)}}{(x - x_n) + 2^{-n}} > \frac{2^{-(n+1)}}{2^{-n} + 2^{-n}} = \frac{1}{4}.$$

□

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Every strongly left-computable number is speedable.

Theorem (Hertling, Hölzl, Janicki (2023))

There exists a strongly left-computable number that is not regainingly approximable.

The converse is not true as the two following known results imply.

Proposition (Merkle, Titov (2020))

Every strongly left-computable number is speedable.

Theorem (Hertling, Hölzl, Janicki (2023))

There exists a strongly left-computable number that is not regainingly approximable.

Corollary

There exists a speedable number that is not regainingly approximable.

However, speedability and regaining approximability are equivalent in the context of nearly computable numbers!

Proposition (Hertling, Hölzl, Janicki (2023))

For a left-computable number x the following are equivalent:

1. *x is regainingly approximable.*
2. *There exists a computable non-decreasing sequence $(x_n)_n$ converging to x and a computable non-decreasing and unbounded function $h\colon \mathbb{N} \to \mathbb{N}$ with $x - x_n < 2^{-h(n)}$ for infinitely many $n \in \mathbb{N}$.*

Proposition (Hertling, Janicki (2023))

For a left-computable number x the following are equivalent:

1. *x is nearly computable.*
2. *For every computable non-decreasing sequence $(x_n)_n$ converging to x, the sequence $(x_{n+1} - x_n)_n$ converges computably.*

Theorem

Let x be a speedable number which is nearly computable. Then x is regainingly approximable.

Proof:
Choose a constant $\rho \in \,]0,1[$ and a computable increasing sequence of rational numbers $(x_n)_n$ converging to x that witness the speedability of x. Since x is nearly computable, the sequence $(x_{n+1} - x_n)_n$ converges computably to zero, as witnessed by some modulus of convergence $f\colon \mathbb{N} \to \mathbb{N}$ that is computable, non-decreasing and unbounded. Define $g\colon \mathbb{N} \to \mathbb{N}$ via

$$g(n) := \begin{cases} 0 & \text{if } f(0) > n, \\ \max\{k \in \mathbb{N} \mid f(k) \leq n\} & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Clearly, g is computable, non-decreasing and unbounded, and we have $x_{n+1} - x_n < 2^{-g(n)}$ for all $n \geq f(0)$.

Fix some $k \in \mathbb{N}$ with $\frac{1}{\rho} \leq 2^k$ and define $h\colon \mathbb{N} \to \mathbb{N}$ via

$$h(n) := \max\{0, g(n) - k\}$$

for all $n \in \mathbb{N}$. Again, h is computable, non-decreasing and unbounded. Let $m := \min\{i \geq f(0) \mid g(i) \geq k\}$ and consider any of the infinitely many $n \geq m$ for which $\frac{x_{n+1} - x_n}{x - x_n} \geq \rho$ holds by choice of ρ and of $(x_n)_n$. For all of these n we have

$$x - x_n \leq \frac{1}{\rho} \cdot (x_{n+1} - x_n) < 2^k \cdot 2^{-g(n)} = 2^{-h(n)}.$$

Hence, x is regainingly approximable. □

Corollary (Main Result 3)

Let x be a left-computable number that is nearly computable. Then the following statements are equivalent:

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Corollary

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Theorem (Stephan, Wu (2005))

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Corollary

There exists a left-computable number which is not speedable and not Martin-Löf random.

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Full-text version: https://arxiv.org/abs/2303.11986

Thanks for your attention!