Computability of generalized graphs

Zvonko Iljazović, Matea Jelić

University of Zagreb, University of Split

September 7, 2023

Introduction

A compact set $S \subseteq \mathbb{R}$ is *semicomputable* if its complement $\mathbb{R} \setminus S$ can be effectively exhausted by rational open intervals.
A compact set $S \subseteq \mathbb{R}$ is *computable* if it is semicomputable and we can effectively enumerate all rational open intervals which intersect S.

Semicomputable and computable sets can be defined in more general spaces - computable metric and topological spaces.
Regardless the ambient space, the following is true:

$$\boxed{S \text{ computable} \Longrightarrow S \text{ semicomputable}}$$

Is it true, then, that

$$S \text{ semicomputable} \Longrightarrow S \text{ computable}$$

holds?

Examples

For example, any semicomputable compact manifold is computable. In fact, any topological semicomputable circle is computable.
Furthermore, if M is manifold with boundary, and M and ∂M are semicomputable, then M is computable.
Also, semicomputable arc with computable endpoints is computable.

Note: Some topological properties force semicomputable set to be computable.

Counterexample

Is it true, then, that

$$S \text{ semicomputable} \Longrightarrow S \text{ computable}$$

holds?

No. In general, a semicomputable set does not have to be computable.

Let γ be a positive real number which is not a computable one. The line segment $[0, \gamma]$ is a **semicomputable** set which is **not computable**. More can be found in *J.S.Miller, Effectiveness for embedded spheres and balls, Electronic Notes in Theoretical Computer Science, 66:127-138, 2002.*

Moreover, computable numbers are dense in every nonempty computable set in $\mathbb{R}$, while there exists a nonempty semicomputable set $S \subseteq \mathbb{R}$ which does not contain any computable number. More can be found in *E.Specker, Der satz von maximum in der rekursiven analysis, In A.HEyting, editor, Constructivity in Mathematics, pages 254-265, North Holland Publ. Comp. Amsterdam, 1959.*

Questions and research directions

So it makes sense to wonder which conditions force a semicomputable set in an arbitrary computable topological space to be a computable one. Actually, we are interested in spaces S such that the following holds:

$$S \text{ semicomputable} \implies S \text{ computable.}$$

Also, we are interested in spaces S such that above implication holds if some subset T of S is semicomputable, as well. For such spaces S and pairs (S, T) - and all homeomorphic to them - we will say that they have *computable type*.

Preliminaries

A triple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space, $\alpha = (\alpha_i)$ is a sequence in X such that $\alpha(\mathbf{N}) \subseteq X$ is dense in (X, d) and such that the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable.

Let $i \in \mathbf{N}$ and $r \in \mathbf{Q}$, $r > 0$. We say that the set $B(\alpha_i, r) = \{x \in X \mid d(x, \alpha_i) < r\}$ is an **(open) rational ball** in a computable metric space (X, d, α). Also, α_i is called a **rational point**.

Now, let $q : \mathbf{N} \to \mathbf{Q}$ be some fixed computable function whose image is the set of all positive rational numbers and let $\tau_1, \tau_2 : \mathbf{N} \to \mathbf{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbf{N}\} = \mathbf{N}^2$. For $i \in \mathbf{N}$ we define

$$I_i = B(\alpha_{\tau_1(i)}, q_{\tau_2(i)}). \tag{1}$$

Sequence $(I_i)_{i \in \mathbf{N}}$ is an effective enumeration of all rational balls in (X, d, α).

Every finite union of rational balls will be called a **rational open set**. Let $\mathbf{N} \to \mathcal{F}(\mathbf{N})$, $j \to [j]$ be some fixed computable function whose range is the set of all nonempty finite subsets of $\mathbf{N}$. For $j \in \mathbf{N}$ we define

$$J_j = \bigcup_{i \in [j]} I_i. \tag{2}$$

Clearly, $(J_j)_{j \in \mathbf{N}}$ is an effective enumeration of all rational open sets in (X, d, α).

Let $S \subseteq X$ be a closed set in (X, d). We say that S is a **computably enumerable** (c.e.) set in (X, d, α) if the set

$$\{i \in \mathbf{N} \mid I_i \cap S \neq \emptyset\}$$

is a c.e. subset of $\mathbf{N}$.

Let $S \subseteq X$ be a compact set in (X, d). We say that S is a **semicomputable** set in (X, d, α) if the set

$$\{j \in \mathbf{N} \mid S \subseteq J_j\}$$

is a c.e. subset of $\mathbf{N}$.

Finally, we say that S is a **computable set** in (X, d, α) if S is both c.e. and semicomputable in (X, d, α).

Computable topological space

Definition

Let $(X, \mathcal{T})$ be a topological space and let (I_i) be a sequence in $\mathcal{T}$ such that the set $\{I_i \mid i \in \mathbf{N}\}$ is a basis for $\mathcal{T}$. A triple $(X, \mathcal{T}, (I_i))$ is called a **computable topological space** if there exist c.e. subsets $C, D \subseteq \mathbf{N}^2$ such that:

- if $i, j \in \mathbf{N}$ are such that $(i, j) \in C$, then $I_i \subseteq I_j$;
- if $i, j \in \mathbf{N}$ are such that $(i, j) \in D$, then $I_i \cap I_j = \emptyset$;
- if $x \in X$ and $i, j \in \mathbf{N}$ are such that $x \in I_i \cap I_j$, then there is $k \in \mathbf{N}$ such that $x \in I_k$ and $(k, i), (k, j) \in C$;
- if $x, y \in X$ are such that $x \neq y$, then there are $i, j \in \mathbf{N}$ such that $x \in I_i$, $y \in I_j$ and $(i, j) \in D$.

Let $(X, \mathcal{T}, (I_i))$ be a fixed **computable topological space**. We define $J_j := \bigcup_{i \in [j]} I_i$.

We say that a closed set S in $(X, \mathcal{T})$ is **computably enumerable** in $(X, \mathcal{T}, (I_i))$ if $\{i \in \mathbf{N} \mid S \cap I_i \neq \emptyset\}$ is a c.e. subset od $\mathbf{N}$.

We say that S is **semicomputable** in $(X, \mathcal{T}, (I_i))$ if S is a compact set in $(X, \mathcal{T})$ and $\{j \in \mathbf{N} \mid S \subseteq J_j\}$ is a c.e. subset of $\mathbf{N}$.

We say that S is **computable** in $(X, \mathcal{T}, (I_i))$ if S is both c.e. and semicomputable in $(X, \mathcal{T}, (I_i))$.

The definition of a semicomputable set (and a computable set) does not depend on the choice of the sequence $([j])_{j \in \mathbf{N}}$.

If (X, d, α) is a computable metric space, then $(X, \mathcal{T}_d, (I_i))$ is a computable topological space where $\mathcal{T}_d$ is a topology induced by the metric d and (I_i) is the sequences defined by (1). Clearly, S is c.e./semicomputable/computable in (X, d, α) if and only if S is c.e./semicomputable/computable in $(X, \mathcal{T}_d, (I_i))$.

Theorem

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space. There exist c.e. subsets $\mathcal{C}, \mathcal{D} \subseteq \mathbf{N}^2$ such that:
- if $i, j \in \mathbf{N}$ are such that $(i, j) \in \mathcal{C}$, then $J_i \subseteq J_j$;
- if $i, j \in \mathbf{N}$ are such that $(i, j) \in \mathcal{D}$, then $J_i \cap J_j = \emptyset$;
- if $\mathcal{F}$ is a finite family of nonempty compact sets in $(X, \mathcal{T})$ and $A \subseteq \mathbf{N}$ is a finite subset of $\mathbf{N}$, then for each $K \in \mathcal{F}$ there is $i_K \in \mathbf{N}$ such that
(i) $K \subseteq J_{i_k}$;
(ii) if $K, L \in \mathcal{F}$ are such that $K \cap L = \emptyset$, then $(i_K, i_L) \in \mathcal{D}$;
(iii) if $a \in A$ and $K \in \mathcal{F}$ are such that $K \subseteq J_a$, then $(i_k, a) \in \mathcal{C}$.

Chainable and circulary chainable continuua

Let X be a set and $\mathcal{C} = (C_0, \ldots, C_m)$ be a finite sequence of subsets of X. We say that $\mathcal{C}$ is a **chain** in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j|,$$

for all $i, j \in \{0, \ldots, m\}$.

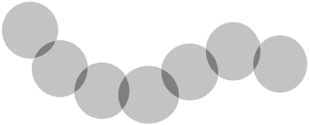

Figure: Chain.

We say that $\mathcal{C}$ is a **circular chain** in X if the following holds:

$$C_i \cap C_j = \emptyset \iff 1 < |i - j| < m,$$

for all $i, j \in \{0, \ldots, m\}$.

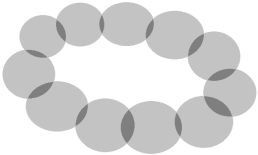

Figure: Circular chain.

Let $A \subseteq X$ and $a, b \in A$. We say that $C_0, \dots, C_m$ **covers** A if $A \subseteq C_0 \cup \dots \cup C_m$, and we say it **covers** A **from** a **to** b if it is also $a \in C_0$ and $b \in C_m$.

Let (X, d) be a metric space. A (circular) chain $C_0, \dots, C_m$ is said to be a ϵ-**(circular) chain**, for some $\epsilon > 0$, if diam $C_i < \epsilon$, for each $i \in \{0, \dots, m\}$ and it is said to be an **open (circular) chain** if every C_i is open in (X, d). In the same way we define the notion of a **compact (circular) chain**.

Let (X, d) be a continuum, i.e. a connected and compact metric space. We say that (X, d) is a **(circulary) chainable continuum** if for every $\epsilon > 0$ there is an open ϵ-(circular) chain in (X, d) which covers X. Suppose $a, b \in X$. We say that (X, d) is a **continuum chainable from** a **to** b if for every $\epsilon > 0$ there is an open ϵ-chain $C_0, \dots, C_m$ which covers X from a to b.

Proposition

(i) Let (X, d) *be a continuum and* $a, b \in X$*. Then* (X, d) *is a chainable continuum from* a *to* b *if and only if for each* $\epsilon > 0$ *there is a compact* ϵ*-chain in* (X, d) *which covers* X *from* a *to* b*.*
(ii) Let (X, d) *be a continuum. Then* (X, d) *is a (circulary) chainable continuum if and only if for each* $\epsilon > 0$ *there is a compact* ϵ*-(circular) chain in* (X, d) *which covers* X*.*

We similarly define the notions of an open and a compact (circular) chain in a topological space.

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. We say that $\mathcal{A}$ **refines** $\mathcal{B}$ if for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Let X be compact, connected and Hausdorff. We say that X is a **(circulary) chainable continuum** if for each open cover $\mathcal{U}$ of X there is an open (circular) chain $C_0, \dots, C_m$ in X which covers X and such that $\{C_0, \dots, C_m\}$ refines $\mathcal{U}$. We similarly define that a continuum is **chainable from** a **to** b.

It follows easily that a metric space (X, d) is a (circulary) chainable continuum if and only if topological space $(X, \mathcal{T}_d)$ is a (circulary) chainable Hausdorff continuum. Also, (X, d) is a continuum chainable from a to b if and only if $(X, \mathcal{T}_d)$ is a Hausdorff continuum chainable from a to b.

Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then is easy to see that X is a (circularly) chainable Hausdorff continuum if and only if Y is a (circularly) chainable Hausdorff continuum. Furthermore, if $a, b \in X$, then X is a Hausdorff continuum chainable from a to b if and only if Y is a Hausdorff continuum chainable from $f(a)$ to $f(b)$.

Computable type

Let A be a topological space. Suppose that the following holds: if $(X, \mathcal{T}, (I_i))$ is a computable topological space and S a semicomputable set in this space such that S and A are homeomorphic, then S is computable. Then we say that A has **computable type**.

Moreover, let A be a topological space and let B be a subspace of A. Suppose that the following holds: if $(X, \mathcal{T}, (I_i)$ is a computable topological space, S and T semicomputable sets in this space and $f : A \to S$ a homeomorphism such that $f(B) = T$, then S is computable. Then we say that (A, B) has **computable type**.

Čičković, E., Iljazović, Z., Validžić, L.: "Chainable and circularly chainable semicomputable sets in computable topological spaces"; Archive for Mathematical Logic, 58 (2019) 885-897.

Proposition

Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and K a semicomputable set in this space which is, as a subspace of $(X, \mathcal{T})$, a continuum chainable from a to b, where a and b are computable points. Then K is a computable set in $(X, \mathcal{T}, (I_i))$.

So, $(K, \{a, b\})$ has computable type if K is continuum chainable from a to b.

Topological graph

Let $n \in \mathbf{N}$ and let $\mathcal{I}$ be a nonempty finite family of (non-degenerate) line segments in $\mathbf{R}^n$ such that the following holds:

$$\text{if } I, J \in \mathcal{I} \text{ are such that } I \neq J \text{ and } I \cap J \neq \emptyset, \text{ then } I \cap J = \{a\},$$

where a is an endpoint of both I and J. Then any topological space G homeomorphic to $\bigcup_{I \in \mathcal{I}} I$ is called a **graph**.

If G is a graph and $x \in G$, we say that x is an **endpoint** of G if there exists an open neighborhood N of x in G such that N is homeomorphic to $[0, \infty)$ by a homeomorphism which maps x to 0.

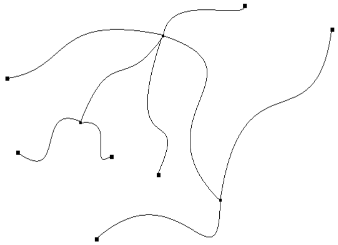

Figure: Graph. The highlighted points are its endpoints.

The following result was proved in *Z.Iljazović, Computability of graphs, Math. Log. Q., 66(1):51-64, 2020.*.

Theorem

Let G be a graph and let E be the set of all endpoints of G. Then (G, E) has computable type.

So, we have we considered spaces more general than graphs, so called *generalized graphs*, and we have generalized previous theorem by showing that an analogue version also holds for generalized graphs.

Generalized graph

Suppose A is a topological space. Let $V \subseteq A$ be a finite subset of A and let $\mathcal{K}$ be a finite family of pairs $(K, \{a, b\})$ where $a, b \in V$, $a \neq b$ and $K \subseteq A$ is a continuum chainable from a to b. Suppose

$$A = V \cup \bigcup_{(K,\{a,b\}) \in \mathcal{K}} K$$

and that the following holds: if $(K, \{a, b\}), (L, \{c, d\}) \in \mathcal{K}$ and $K \neq L$, then $\text{card}(K \cap L) < \aleph_0$.
Then the triple $(A, \mathcal{K}, V)$ is called a **generalized graph**.

Let $(A, \mathcal{K}, V)$ be a generalized graph and let $a \in V$. We say that a is an **endpoint** of $(A, \mathcal{K}, V)$ if there exist only one $K \subseteq A$ and at least one $b \in V$ such that $(K, \{a, b\}) \in \mathcal{K}$.

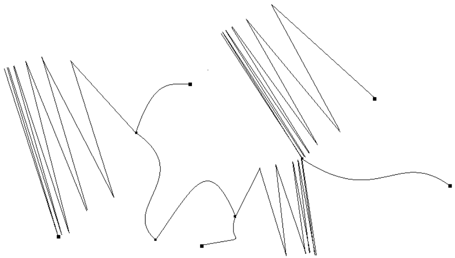

Figure: Generalized graph.

Let G be a graph, let $\mathcal{I}$ be the family from the definition of G and let f be the homeomorphism from the definition of G. Let E be the set of all endpoints of all $I \in \mathcal{I}$. Let $\mathcal{K}$ be the family of all pairs of form $(f(I), \{f(a), f(b)\})$ such that $I \in \mathcal{I}$ and a and b are endpoints of I. Finally, let $V = f(E)$. Then $(G, \mathcal{K}, V)$ is obviously a generalized graph. We have that x is an endpoint of G if and only if x is an endpoint of $(G, \mathcal{K}, V)$.

Let

$$K = (\{0\} \times [-1,1]) \cup \left\{ \left(x, \sin\frac{1}{x} \right) \mid 0 < x \leq 1 \right\}.$$

Let $a = (0, -1)$ and $b = (1, \sin 1)$.
It is known that K is a continuum chainable from a to b.
A triple $(K, \{(K, \{a, b\})\}, \{a, b\})$ is a generalized graph.
But K is not a graph: K is not locally connected and each graph is easily seen to be locally connected.

Let $c = (0, 1)$. Since K is also a continuum chainable from b to c, triple $(K, \{(K, \{a, c\}), (K, \{b, c\})\}, \{a, b, c\})$ is a generalized graph.
One can easily obtain that $(K, \{a, b, c\})$ has computable type.

So, we have proven then following result:

Theorem

If $(A, \mathcal{K}, V)$ is a generalized graph and B is the set of all its endpoints, then (A, B) has computable type.

It is easy to conclude the following: if $(A, \mathcal{K}, V)$ is a generalized graph, B the set of all its endpoints, A' a topological space and $f : A \to A'$ a homeomorphism, then

$$(A', \mathcal{K}', V'),$$

$\mathcal{K}' = \{(f(K), \{f(a), f(b)\}) \mid (K, \{a, b\}) \in \mathcal{K}\}$, $V' = f(V)$, is a generalized graph and $f(B)$ is the set of all its endpoints.

In view of previous remark, it is enough to prove the following proposition.

Proposition

Let $(X, \mathcal{T}, (I)_i)$ be a computable topological space and let S and T be semicomputable sets in this space. Suppose there are $\mathcal{K}$ and V such that $(S, \mathcal{K}, V)$ is a generalized graph and T is the set of all its endpoints. Then S is computable.

Use of the following Lemma was of a great importance for proving the main result.

Lemma

Let (K, d) *be a continuum chainable from* a *to* b, $a, b \in K$ *and let* $c \in K$ *be arbitrary. For each* $\epsilon > 0$ *there is a nontrivial continuum* $L \subseteq K$ *such that* $c \in L$ *and* $L \subseteq B(c, \epsilon)$.

And prove this Lemma, we had to distinguish two cases: when $c = a$ or $c = b$ and when $c \neq a, b$.

If $c = a$ or $c = b$, the conclusion arises from the following:

Lemma

Let (K, d) be a continuum chainable from a to b, $a, b \in K$. Let $\epsilon > 0$ be arbitrary. Then there exist $c \in K$ and $L \subseteq K$ such that $c \neq a$, L is a continuum chainable from a to c and $L \subseteq B(a, \epsilon)$.

Crutial for proving it was the following well-known fact.

Lemma

Let (X, d) be a compact metric space. Let $(\mathcal{C}^i)$, where $\mathcal{C}^i = (C_0^i, \dots, C_{m_i}^i)$, $i \in \mathbb{N}$, be a sequence of chains such that $\overline{C_0^{i+1}}, \dots, \overline{C_{m_{i+1}}^{i+1}}$ strongly refines $C_0^i, \dots, C_{m_i}^i$ and such that $diam(C_j^i) < 2^{-i}$, for each $i \in \mathbb{N}$ and for each $j \in \{0, \dots, m_i\}$. Let

$$S = \bigcap_{i \in \mathbb{N}} (\overline{C_0^i} \cup \cdots \cup \overline{C_{m_i}^i}).$$

Then S is a continuum chainable from a to b, where $a \in \bigcap_{i \in \mathbb{N}} C_0^i$, $b \in \bigcap_{i \in \mathbb{N}} C_{m_i}^i$.

We have shown the similar one to prove the case $c \neq a, b$.

Lemma

Let (X, d) be a compact metric space and let $(\mathcal{C}^i)$, where $\mathcal{C}^i = (C_0^i, \dots, C_{m_i}^i)$, $i \in \mathbb{N}$, be a sequence of open chains such that $diam(C_j^i) < 2^{-i}$, for each $i \in \mathbb{N}$ and for each $j \in \{0, \dots, m_i\}$ and such that $(\overline{C_0^i} \dots, \overline{C_{m_i}^i})$ is also a chain and $\overline{C_0^{i+1}}, \dots, \overline{C_{m_{i+1}}^{i+1}}$ refines $C_0^i, \dots, C_{m_i}^i$, for each $i \in \mathbb{N}$.
Let

$$S = \bigcap_{i \in \mathbb{N}} \left(\overline{C_0^i} \cup \dots \cup \overline{C_{m_i}^i} \right).$$

Then S is nonempty, connected and compact (i.e. nonempty continuum).

By the the definition of a generalized graph, we let its edges to intersect in multiple points as long as there are finitly many of them. A question arises naturally - if we allow infinite intersections of edges, can we state an analogous result for such an object. The answer is negative. Namely, there is continuum L_1 in $\mathbb{R}^2$ chainable from $(0, 1)$ to $(1, 1)$ which intersects continuum L_2 chainable from $(0, -1)$ to $(1, -1)$ in inifinitly many, yet countable, points. Union $L_1 \cup L_2$ is semicomputable set which is not computable.
More in *M. Čelar, Z.Iljazović, Computability of glued manifold, Journal of Logic and Computatiuon, 32(1):65-97, 2022.*