Continuity and computability: Transition mechanism and admissible extension

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1 The origin of our concern

The main theme: an epistemological consideration on ‘some computable functions in an extended sense’

The universe of discourse: real numbers, real number sequences, real total functions

Euclidean topology on the real line

Assumed: traditional theory of computable (continuous) functions

Urge to study computability aspects of some functions which are not necessarily continuous

Methods: mathematical theories such as Fine spaces, uniform topological spaces, Fréchet spaces etc., endowed with 'computability strucructures'

(Following Pour-El and Richards)

$$\downarrow$$

The theory of 'irrational-based (IB-) computability': a unifying conception of computability for the functions we had treated

The conventional computability of a function: an effectivization of 'locally uniform (LU-) continuity'

To effectivize 'continuity' ('location-wise (LW-) continuity')

$$\downarrow$$

'LW-computability'

Current concerns:

the transition process from continuity to computability;

admissibility of the extension from LW-computability to IB-computability

2 Continuity and computability: definitions

Definition 2.1 1) (LW-continuity)

$$\forall x \forall \varepsilon > 0 \exists \delta > 0$$
$$\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

2) (IB-continuity)

I: the set of all irrational numbers.

A function f is IB-continuous if:

$$\forall x \in \mathbf{I} \forall \varepsilon > 0 \exists \delta > 0$$
$$\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Definition 2.2 (LW-computability) A real function f is *location-wise computable* (LW-computable) if:

(i) (Sequential computability) For every computable sequence of real numbers, $\{x_m\}$, $\{f(x_m)\}$ is a computable sequence of real numbers.

(ii) (LW-effective (LWE-)continuity) For each computable sequence of real numbers, say $\{x_m\}$, there is a recursive function, say δ, satisfying

$$\forall m \forall p \forall y. |x_m - y| < \frac{1}{2^{\delta(m,p)}} \Rightarrow$$

$$|f(x_m) - f(y)| < \frac{1}{2^p}. \quad (1)$$

Definition 2.3 (IB-computability: [2]) A function f is *irrational-based (IB-) computable* if:
(i) (IB-sequential computability) For any computable *irrational* sequence $\{x_m\}$, $\{f(x_m)\}$ is computable.

(ii) (IB-effective (IBE-) continuity) For any $\{x_m\}$ as above, there is a recursive function $\delta(m,p)$:

$$\forall m \forall p \forall y. |x_m - y| < \frac{1}{2^{\delta(m,p)}} \Rightarrow$$

$$|f(x_m) - f(y)| < \frac{1}{2^p}.$$

3 Some consequences

Proposition 3.1 1) LW-continuity implies IB-continuity

2) Computability implies LW-computability.

3) LW-computability implies IB-computability.

4) Many examples of IB-computable but *not* LW-computable functions (cf. [2])

5) An example of uniformly continuous, not computable, LW-computable function

An example of discontinuous, LW-computable function

6) LW-computability is closed under composition

7) The patching theorem holds for LW-computable functions

8) The set of discontinuous points of an LW-computable function is nowhere dense.

Remark 1 The family of LW-computable functions is closed under 'LWE-convergence' of 'LWAQ-computable sequences'.

4 Admissible extension

Definition 4.1 (Conjunctive principle and admissible extension: cf. [1])

(1) The 'conjunctive principle' of concept extension of a mathematical theory consists of the three conditions below.

1° Inner necessity

2° Substantial conservation

3° Formal conservation

(2) An extension of a mathematical theory is 'admissible' if the reslut-ing theory satisfies the conjunctive principle

Theorem 1 IB-continuity is an admissible extension of LW-continuity.

Proof $\mathbf{J}$: a set of real numbers $C(\mathbf{J}, f)$:

$$\forall x \in \mathbf{J} \forall \varepsilon > 0 \exists \delta > 0$$

$$\forall y. |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

If $\mathbf{J} = \mathbf{R}$, then $C(\mathbf{J}, f)$ asserts LW-continuity of f

If $\mathbf{J} = \mathbf{I}$, then it asserts IB-continuity of f

So, LW-continuity and IB-continuity share the same law $C(\mathbf{J}, f)$, or 3°

1°: Urge to study some interesting functions, discontinuous at some rational points, exhibiting certain features of continuity

2°: An IB-continuous function continuous at rational numbers too is LW-continuous

Theorem 2 IB-computability is an admissible extension of LW-computability

Proof

$\mathcal{A}$: a set of sequences from $\mathbf{J}$

$D(\mathcal{A}, f)$: For each sequence $\{x_m\} \in \mathcal{A}$,

(i) The sequence $\{f(x_m)\}$ is computable

(ii) There is a recursive function, δ, satisfying

$$\forall m \forall p \forall y. |x_m - y| < \frac{1}{2^{\delta(m,p)}} \Rightarrow$$
$$|f(x_m) - f(y)| < \frac{1}{2^p}$$

In the initial theory:

$\mathcal{A}$ is the family $\mathcal{R}$ of all computable real sequences

$D(\mathcal{A}, f)$: f is LW-computable

In the new theory:

$\mathcal{A}$ is the family $\mathcal{I}$ of all computable irrational sequences

$D(\mathcal{A}, f)$: f is IB-computable

The formal conservation 3°

5 Transition mechanism: from continuity to computability

$\mathbf{J} = \mathbf{R}/\mathbf{I}$ ε: a rational number

$f : A \mapsto B; C$: "f assigns (a single value) B to each A, and C holds for B"

f is a function and is $\mathbf{J}$-continuous::

(a) $f : x \mapsto f(x)$

(b) $f : x \in \mathbf{J}, \varepsilon > 0 \mapsto$

$$\delta = \delta_{x,\varepsilon} > 0 \;;$$

$\forall y.|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon (*)$

The image of transition from this to $\mathbf{J}$-computability of f

$\mathcal{C}$: the set of computable real numbers

(c) '$x \in \mathcal{C} \cap \mathbf{J} \;\mapsto\; f(x)$'
is *effective*

(d) '$x \in \mathcal{C} \cap \mathbf{J}, \varepsilon > 0 \;\mapsto\; \delta = \delta_{x,\varepsilon} > 0$'
is *effective*

To give a mathematical expression to (c) and (d)

$\mathcal{R}$: the family of all computable sequences from $\mathcal{C} \cap \mathbf{J}$

$\mathcal{B}$: the family of all binary recursive functions

Definition 5.1 Transition mechanism from continuity to computability: transformation from (a) and (b) to (i) and (ii)

(i) $f : \{x_m\} \in \mathcal{R} \mapsto$

$$\{f(x_m)\} \in \mathcal{R}$$

(ii) $f : \{x_m\} \in \mathcal{R} \mapsto \delta \in \mathcal{B}$;

$$\forall m, p \forall y. |x_m - y| < \frac{1}{2^{\delta(m,p)}}$$

$$\Rightarrow |f(x_m) - f(y)| < \frac{1}{2^p}$$

Similarly with the transition from IB-continuity to IB-computability.

6 Appendix I

Definition 6.1 (LWAQ-computability of function sequence) A function sequence $\{f_n\}$ is called *location-wise asymptotically equi-computable* (LWAQ-computable) if it safisfies the following 1° and 2°.

Let $\{x_m\}$ be a computable real sequence.

1°. (Sequential computability) $\{f_n(x_m)\}$ is computable as a double sequence of real numbers.

2°. (LW-effective asymptotical equicontinuity: LWEAQ-continuity) There are recursive functions $\beta_1(m,p)$ and $\beta_2(m.p)$ such that

$$\forall n \geq \beta_1(m,p) \forall y . |x_m - y| < \frac{1}{2^{\beta_2(m,p)}} \Rightarrow$$

$$|f_n(x_m) - f_n(y)| < \frac{1}{2^p}.$$

β_2 may be called a recursive asymptotic modulus of continuity (for $\{f_n\}$ and) $\{x_m\}$.

Definition 6.2 (LWE-convergence) A function sequence $\{f_n\}$ will be said to *location-wise effectively converge* (LWE-converge) to f if the following condition holds.

For any $\{x_m\}$, there are recursive funtions α_1 and α_2 such that

$$\forall n \geq \alpha_1(m,p) \forall y. |y - x_m| < \frac{1}{2^{\alpha_2(m,p)}} \Rightarrow |f_n(y) - f(y)| < \frac{1}{2^p}.$$

Theorem 3 (Closure property) If $\{f_n\}$ is an LWAQ-computable sequence (Definition 6.1) and LWE-converges to f (Definition 6.2), then f is LW-computable.

7 Appendix II

Example of concept extension

T_1: the theory of positive integers with the operation $+$

$$\downarrow$$

a requirement to solve an equation like $n+x=m$ for x regardless of the magnitutes of n and m

$$\downarrow$$

Creation of a new domain of all integers and the new operation $-$ (subtraction)

$$\downarrow$$

T_2: the theory of integers with $+$ and $-$

References

[1] Mariko Yasugi, *Two types of concept extension in mathematics*, Dissertation (Kyoto University), 2014. (in Japanese) Available at: http://hdl.handle.net/2433/188398 DOI 10.14989/doctor.k17995

[2] Mariko Yasugi, Yoshiki Tsujii, Takakazu Mori, *Irrational-based computability of functions*, Advances in Mathematical Logic: SAML 2018, Kobe; Springer (2021), 181-204. Available also at

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