Computability of irreducible continua

Tea Arvaj and Zvonko Iljazović

University of Zagreb

A topological pair (A,B) of space A and its subspace B is said to have computable type if, whenever $f : A \to X$ is an embedding of A in a computable metric space such that $f(A)$ and $f(B)$ are semicomputable, then $f(A)$ is computable.

It is known that $([0,1],\{0,1\})$ has computable type. In other words, an arc together with its endpoints has computable type.

However, it is known that a more general result holds:
if K is a continuum chainable from a to b, then $(K,\{a,b\})$ has computable type. That a continuum (i.e. a connected and compact metric space) K is chainable from a to b means that for every $\varepsilon > 0$ there exist finitely many open sets $C_0,\dots,C_n$ in K whose diameters are less than ε, which cover K, such that $a \in C_0$, $b \in C_n$, and $C_i \cap C_j = \emptyset$ iff $|i-j| > 1$.

If A is an arc with endpoints a and b, then A is a continuum chainable from a to b. On the other hand, a continuum K chainable from a to b need not be an arc (for example topologist's sine curve).

In this talk we examine more general spaces, namely irreducible continua. A continuum K is irreducible between a to b, where $a,b \in K$, if there exist no proper sobcontinua of K that contain both a and b.

We have the following result.

Theorem 1. *Let K be a continuum irreducible between a and b. Then the pair* $(K,\{a,b\})$ *has computable type.*

It is known that every computable set contains a dense subset of computable points. So a question naturally arises: if K is a semicomputable continuum irreducible between a and b, where a and b are not necesserily computable (then K is also not necessarily computable), does K contain at least one computable point.

Answer to that question is affirmative in the case when K is an arc, which is shown in [4].

We show that the previuos result also holds for irreducible continua, with an additional assumption that the continuum K can be shown as a union of its three proper subcontinua such that no two of them cover K. Furthermore, in that case K not only contains a computable point, but contains a computably enumerable subset with nonempty interior in K.

References

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