Computability of One-Point Metric Bases

Konrad Burnik * Zvonko Iljazović † Lucija Validžić ‡

A connection between the computability of an arc and the computability of its endpoints has been very well studied. Miller has shown in [4] that there exists a computable arc in $\mathbf{R}^2$ with noncomputable endpoints. However, a computable arc in $\mathbf{R}$ has to be of the form $[a, b]$, where a and b are computable real numbers. On the other hand, it is known that if endpoints of a semicomputable arc are computable, then the arc is computable ([1, 5]).

Endpoints of a segment in $\mathbf{R}$ have an interesting property. Namely, if x_0 is an endpoint, than any other point in the segment is uniquely determined by its distance from x_0, i.e. we have that $d(x, x_0) = d(y, x_0)$ implies $x = y$. In a general metric space (X, d), we say that a point with such property is a *metric basis* for (X, d) [3]. Using this notion, we generalise the result for computable arcs in $\mathbf{R}$ to computable metric spaces. Here the assumption of *effective compactness* of the underlying computable metric space (X, d, α) plays an important role.

Theorem 1. *Assume that (X, d, α) is effectively compact computable metric space such that the space (X, d) has finitely many connected components. If $x_0 \in X$ is metric basis for (X, d), then x_0 is a computable point in (X, d, α).*

That (X, d, α) is effectively compact means that (X, d) is compact and that there is a computable function $\varphi : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$

$$X = \bigcup_{i=0}^{\varphi(k)} B(\alpha_i, 2^{-k}).$$

Effective compactness of (X, d, α) is actually equivalent to X being a computable set in (X, d, α).

If $a, b \in \mathbf{R}^n$ then a and b are metric bases for the line segment $\overline{ab}$. So, Theorem 1 easily implies that a computable segment in $\mathbf{R}^n$ has computable endpoints.

Example 1. The assumption of effective compactness in Theorem 1 cannot be omitted. Assume that γ is a positive real number which is left computable but not computable. The segment $[0, \gamma]$ can be viewed as a subspace of the standard computable metric space on $\mathbf{R}$ ([2]). Obviously γ is a metric basis for $[0, \gamma]$, but it is not computable in $[0, \gamma]$. However, in this example 0 is also a metric basis, so at least there exists a computable metric basis.

*Xebia Data, email: kburnik@gmail.com
†University of Zagreb, email: zilj@math.hr
‡University of Zagreb, email: lucija.validzic@math.hr

We can look at the segment $[-\gamma, \gamma]$. Using dense computable sequence in $[0, \gamma]$ we can easily construct a dense computable sequence in $[-\gamma, \gamma]$, so this segment can also be viewed as a subspace of the standard computable metric space on $\mathbf{R}$. However, here neither of the points $-\gamma$, γ (which are only metric bases) are not computable.

The results of Theorem 1 can be extended to some non-compact cases. We have proved the following.

Theorem 2. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Suppose (X, d) is homeomorphic to $[0, +\infty\rangle$ (i.e. (X, d) is a topological ray). If x_0 is a metric basis for (X, d) then x_0 is a computable point in (X, d, α).*

References

[1] Z. Iljazović. Co-c.e. spheres and cells in computable metric spaces. *Logical Methods in Computer Science*, 7(3:05):1–21, 2011.

[2] Z. Iljazović and L. Validžić. Maximal computability structures. *Bulletin of Symbolic Logic*, 22(4):445–468, 2016.

[3] R.A. Melter and I. Tomescu. Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, 25:113–121, 1984.

[4] J.S. Miller. Effectiveness for embedded spheres and balls. *Electronic Notes in Theoretical Computer Science*, 66:127–138, 2002.

[5] E. Čičković, Z. Iljazović, and L. Validžić. Chainable and circularly chainable semicomputable sets in computable topological spaces. *Archive for Mathematical Logic*, 58:885–897, 2019.