Subrecursive degrees of representations of irrational numbers outside the cone of Cauchy sequences

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Abstract. We consider different representations of irrational numbers: Cauchy sequences, Dedekind cuts, continued fractions, etc. Any one of them can be computably transformed into any other, but we are interested in the complexity of these transformations. It has been known since the early days of computable analysis that a computable irrational number may possess representations of widely varying complexity (see [6,11,12]).

Let R_1 and R_2 be two representations of irrational numbers. We say that R_1 is subrecursive in R_2, denoted $R_1 \preceq_S R_2$, if given any R_2-representation f of an irrational number $\alpha \in (0,1)$, we can compute an R_1-representation g of the same α by an algorithm, which uses f as oracle but with *no unbounded search allowed*. For example, let $\mathcal{C}$ be the representation by Cauchy sequences (converging with speed $(n+1)^{-1}$) and $\mathcal{D}$ be the representation by Dedekind cuts. Then $\mathcal{C} \preceq_S \mathcal{D}$, but $\mathcal{D} \not\preceq_S \mathcal{C}$. For example, one can construct an irrational number α with complex base-2 expansion, but with elementary base-3 expansion. Such an α has an elementary Cauchy sequence, but its Dedekind cut cannot be elementary.

The relation $\preceq_S$ induces (in the usual way) a degree structure on the set of representations of irrational numbers. The study of this structure was systematically introduced in [8,9] and has since been a constantly active area of research, see [1,2,3,4,5,7,10].

The structure turns out to be a distributive lattice with respect to conjunction and disjunction of representations (as usually defined in computable analysis, see Chapter 3 in [13]). The degree of the representation $\mathcal{W}$ (a sequence of open intervals with rational endpoints, converging to α) is the least element in the structure, which follows from the fact that every computable number has a $\mathcal{W}$-representation, which is elementary.

Thus we have $\mathcal{W} \preceq \mathcal{C}$, but a diagonalization argument shows that $\mathcal{C} \not\preceq_S \mathcal{W}$ and therefore $\mathcal{W}$ is strictly subrecursive in $\mathcal{C}$. Up to this point the degree of $\mathcal{W}$ was the only known subrecursive degree, which lies outside the cone of the degree of $\mathcal{C}$.

The aim of this talk is to present some other such examples, based on trace functions.

A function $T_\uparrow : \mathbb{Q} \to \mathbb{Q}$ will be called *a trace function from below* for the irrational number α if $q < \alpha$ implies $q < T_\uparrow(q) < \alpha$ for any $q \in \mathbb{Q}$.

A function $T_\downarrow : \mathbb{Q} \to \mathbb{Q}$ will be called *a trace function from above* for the irrational number α if $q > \alpha$ implies $q > T_\downarrow(q) > \alpha$ for any $q \in \mathbb{Q}$.

The trace functions from below or from above do not define a representation, because they do not determine α uniquely, but we will combine them with the other representations. For any representation R, let $R_\uparrow$ be the conjunction of R with a trace function from below, let $R_\downarrow$ be the conjunction of R with a trace function from above and $R_{\uparrow\downarrow}$ be the conjunction of $R_\uparrow$ and $R_\downarrow$.

For example, it is not hard to see that $\mathcal{C}_\uparrow$ ($\mathcal{C}_\downarrow$) has the same degree as the general sum approximation from below (above), see [9], which has the same degree as the dual (standard) Baire sequences, see [7]. The degree of $\mathcal{C}_{\uparrow\downarrow}$ is the same as the degree of the continued fraction.

But combined with $\mathcal{W}$ we obtain three new degrees of $\mathcal{W}_{\uparrow}$, $\mathcal{W}_{\downarrow}$ and $\mathcal{W}_{\uparrow\downarrow}$, which turn out to be different and incomparable with $\mathcal{C}$.

Moreover, their disjunctions with $\mathcal{C}$ are not subrecursive in $\mathcal{W}$ and thus we have examples of degrees, which lie strictly between the degree of $\mathcal{W}$ and the degree of $\mathcal{C}$. An open question remains, whether the disjunctions of $\mathcal{C}$ with $\mathcal{W}_{\uparrow}$ and $\mathcal{W}_{\downarrow}$ are subrecursively incomparable.

Keywords: representations of irrational numbers, subrecursive reducibility, cone of Cauchy sequences, trace functions from below and from above

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