Computable intersection points of circularly chainable continua

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Abstract

Let us consider a computable function $f : [0,1] \to \mathbb{R}$ such that $f(0) < 0$, $f(1) > 0$. A well known result ([2]) is that f has a computable zero. If we let $S = \mathbb{R} \times \{0\}$, $\Gamma(f) = \{(x, f(x)) : x \in [0,1]\}$ the result can then be paraphrased in this way: $\Gamma(f) \cap S$ contains a computable point. We note that S and $\Gamma(f)$ are computable subsets of $\mathbb{R}^2$. Also note that $S = \mathbb{R}^2 \setminus (U \cup V)$, where $U = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper, and $V = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ the lower half-plane. The graph $\Gamma(f)$ intersects both U and V, and is both connected and compact (i. e. a *continuum*) considering the Euclidean metric.

Let (X, d, α) be a computable metric space, with a computable set K and disjoint computably enumerable open sets U and V in this space such that K intersects both U and V. Some conditions under which the set $K \cap S$ contains a computable point, where $S = X \setminus (U \cup V)$, have been examined in [1]. One such condition is that K is an arc. An arc is a paradigmatic example of a *chainable* continuum, while a topological circle is a paradigmatic example of a *circularly chainable* continuum.

Definition 1. Let (K, d) be a metric space A finite sequence $C_0, \dots, C_m$ of nonempty open subsets of K is said to be a chain in K if for all $i, j \in \{0, \dots m\}$

$$|i - j| > 1 \iff C_i \cap C_j = \emptyset.$$

$C_0, \dots, C_m$ is said to be a circular chain in K if for all $i, j \in \{0, \dots m\}$

$$1 < |i - j| < m \iff C_i \cap C_j = \emptyset.$$

Let $\epsilon > 0$. If $\operatorname{diam}(C_i) < \epsilon$ for all $i \in \{0, \dots, m\}$, we say that $C_0, \dots, C_m$ is an ϵ-(circular) chain.

Definition 2. Let (K, d) be a continuum. We say that (X, d) is a (circularly) chainable continuum if for each $\epsilon > 0$ there exits an ϵ-(circular) chain $C_0, \dots, C_m$ in (K, d) which covers K.

It follows that further generalizations naturally arise if we consider a continuum K, either chainable or circularly chainable. Concretely, as shown in [1], if K is a chainable continuum and $K \cap S$ is totally disconnected, this again results in existence of a computable point in $K \cap S$.

In this talk we consider K to be a circularly chainable continuum. We focus primarily on a case in which K is not chainable. In such a case we can further weaken conditions on K, as semi-computability of K implies its computability. We prove that, if $K \cap S$ is totally disconnected, it contains a computable point.

References

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