DEGREES OF UNSOLVABILITY OF NATURAL PROBLEMS: A REALIZABILITY-THEORETIC APPROACH

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The theories of *degrees of unsolvability* and *realizability interpretation* both have long histories, having both been born in the 1940s. S. C. Kleene was a key figure who led the development of both theories. Despite having been developed by the same person, there seems to have been little deep mixing of these theories until recently. In this talk, we will reconstruct the theory of degrees of unsolvability from the perspective of realizability theory.

In classical computability theory, the notion of degrees of unsolvability for *natural* arithmetical decision problems only plays a role in counting the number of alternations of quantifiers. Interestingly, however, when the realizability interpretation is combined with many-one reducibility, it becomes possible to classify *natural* decision problems in a very nontrivial way. Note here that, obviously, a (decision) problem is described by a formula, not a set.

Definition. An arithmetical formula φ is *many-one reducible to* ψ if there exists a computable function h such that any realizer for $\varphi(x)$ is effectively converted into a realizer for $\psi(h(x))$ and vice versa.

Alternatively, we may consider a mathematical universe based on the realizability interpretation, which is the category of represented spaces/modest sets or assemblies or the realizability topos. A subobject in such a category is interpreted as a subset with witnesses. Considering the preorder on subobjects induced by pullback, we obtain the same notion as the above many-one reducibility [3].

Using this notion of reducibility, for example, natural Σ_2-decision problems are classified as follows [3]:

- Boundedness for posets is $\forall^\infty$-complete.
- Finiteness of width for posets is $\forall^\infty\forall$-complete.
- Non-density for linear orders is $\exists\forall$-complete.

Using a stronger reduction makes it possible to classify natural decision problems in more detail.

Definition. An arithmetical formula φ is *many-one bi-reducible to* ψ if both φ and its dual are many-one reducible to ψ and its dual via a common h.

In fact, all of the above examples are bicomplete w.r.t. the corresponding classes. Using this notion, for example, natural Π_3-decision problems are classified as follows [2, 1]:

- Being lattice for posets is $\forall\forall^\infty$-bicomplete.

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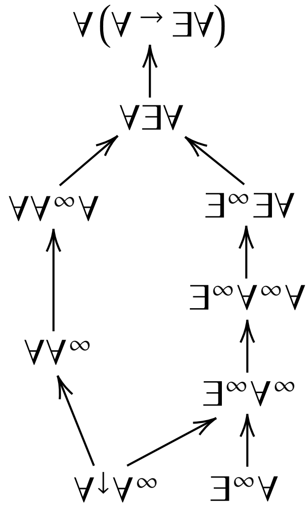

- Atomicity for posets is $\forall\forall^\infty$-bicomplete.
- Local finiteness for graphs is $\forall\forall^\infty\forall$-bicomplete.
- Complementedness for posets is $\forall\exists\forall$-bicomplete.
- Infiniteness of width for enumerated posets is $\exists^\infty\exists\forall$-bicomplete.
- Cauchyness for rational sequences is $\forall^\downarrow\forall^\infty$-bicomplete.
- Simple normality in base 2 for real numbers is $\forall^\downarrow\forall^\infty$-bicomplete.
- Perfectness for binary trees is $\forall(\forall \to \exists\forall)$-bicomplete.

In the above figure, $\mathsf{P} \to \mathsf{Q}$ means that a P-complete problem is many-one bireducible to a Q-complete problem. No further arrows can be added [2, 1]. By the 1950s, all the tools were in place. If the development of computability theory had been slightly different, these results could have been announced in the 1950s (probably by Kleene).

Indeed, in computational complexity theory, the combination of polytime many-one reducibility and realizability interpretation was already being studied in the 1970s under the name of *Levin reducibility*, so it is a little strange that a similar notion was not being studied in computability theory. In the context of intuitionistic mathematics, related research has been conducted by Veldman [4], which in fact is what triggered our study.

References

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