Towards Numerical Stability Analysis via Universal Envelopes

Eike Neumann
Swansea University, UK

The most prevalent paradigm by far for computation with continuous data is to work with finite approximations of infinite objects to a fixed accuracy and to perform all operations approximately. In this context, numerical stability analysis becomes a central concern. In [Neu22a] we have connected questions of numerical stability to questions about (universal) envelopes [Neu19]. More precisely, for a function $f: X \to Y$ between computable metric spaces, let

$$^{\dagger}f : X \times (0, +\infty) \rightrightarrows Y,\ ^{\dagger}f(x, \delta) = f(B(x, \delta)), \tag{1}$$

be the *backwards approximation* of f, representing an approximate evaluation of f to a given finite accuracy. One is lead to the question whether we have convergence

$$^{\dagger}f_n(\cdot, \delta) \circ \cdots \circ {}^{\dagger}f_1(\cdot, \delta)(x) \to f_n \circ \cdots \circ f_1(x) \qquad \text{as } \delta \to 0. \tag{2}$$

In [Neu22a, Theorem 1] we have given a necessary and sufficient condition for (2) using certain envelopes of the functions f_i. This criterion is satisfactory provided that all functions appearing in (2) are *uniformly envelopable* [Neu22b]. It is natural to ask whether this result can be extended to more general classes of functions.

In some situations it is possible to obtain approximations with stronger guarantees. Here, we consider functions $f: X \to R$ between computable metric spaces with an isometric open embedding $j: R \to Y$ such that for some $\Delta\colon (0, \eta) \to (0, +\infty)$ with $\Delta(\delta) \leq \delta$, the function

$$^{\dagger}_{\Delta}f : X \times (0, \eta) \rightrightarrows Y,\ ^{\dagger}_{\Delta}f(x, \delta) = j \circ f(B(x, \delta)) \setminus B(Y \setminus j(R), \Delta(\delta)) \tag{3}$$

has non-empty values everywhere. Informally, we associate with a function between computable metric spaces a "formal range" R, and require approximate solutions calculated by $^{\dagger}_{\Delta}f$ to be uniformly bounded away from the complement of the formal range.

As a simple motivating example, consider the problem of labelling a pair of numbers with equality information:

$$\text{equal?}\colon \mathbb{R}^2 \to \left(\mathbb{R}^2 \times \{0\} + \Delta_{\mathbb{R}} \times \{1\}\right),\ \text{equal?}(x, y) = \begin{cases} (x, y, 1) & \text{if } x = y \\ (x, y, 0) & \text{if } x \neq y. \end{cases} \tag{4}$$

This function is not uniformly envelopable. The multi-valued function $^{\dagger}\text{equal?}(x, y, \delta)$ is allowed to output $(x', y', 0)$ with $|x' - y'|$ arbitrarily close to zero. Choosing the actual range

$$R = \left(\mathbb{R}^2 \setminus \Delta_{\mathbb{R}}\right) \times \{0\} + \Delta_{\mathbb{R}} \times \{1\}$$

as the "formal range", the (well-defined) function $^{\dagger}_{\Delta}\text{equal?}(x, y, \delta)$ with $\Delta(\delta) = \delta$ will either output $(x', x', 1)$ or $(x', y', 0)$ with $|x' - y'| \geq \delta$. Generalisations of (4) include problems such as labelling eigenvalues of a matrix with their multiplicity, an important problem in computable linear algebra [ZB04].

We are lead to the question: given functions $f_i : X_i \to R_i$ with isometric embeddings $j_i : R_i \to X_{i+1}$, when do we have convergence

$$^{\dagger}f_n(\cdot, \Delta_{n-1} \circ \cdots \circ \Delta_1(\delta)) \circ \cdots \circ {}^{\dagger}_{\Delta_2}f_2(\cdot, \Delta_1(\delta)) \circ {}^{\dagger}_{\Delta_1}f_1(x, \delta) \to j_n \circ f_n \circ j_{n-1} \circ \cdots \circ f_2 \circ j_1 \circ f_1(x) \quad \text{as } \delta \to 0\ ? \tag{5}$$

To connect this problem to universal envelopes, we assume that each function f_i has a universal envelope of the shape $F_i : X_i \to \mathscr{K}_{\perp}\left(\overline{X}_{i+1}\right)$ with inclusion map $k_i \circ j_i$, where $k_i : Y \to \overline{Y}$ is a uniformly continuous open embedding into a computable metric space $\overline{Y}$. While not all functions between computable metric spaces have this property, it is satisfied for all uniformly envelopable (and thus all continuous) functions and all open functions. Let $(F_i \circ j_i)/j_i$ denote the greatest continuous extension of $F_i \circ j_i$ to X_i along j_i [Esc98], and let $((F_i \circ j_i)/j_i)/k_i$ denote the greatest continuous extension of $(F_i \circ j_i)/j_i$ to $\overline{X}_i$ along k_i. We obtain the following sufficient criterion for convergence:

Theorem 1. Let $f_1, \ldots, f_n$ be functions as above, and let $F_i : X_i \to \mathscr{K}_{\perp}\left(\overline{X}_{i+1}\right)$ be a universal envelope of f_i for all i. If

$$(((F_n \circ j_{n-1})/j_{n-1})/k_{n-1})_* \circ \cdots \circ (((F_2 \circ j_1)/j_1)/k_1)_* \circ F_1(x) = \{k_n \circ j_n \circ f_n \circ j_{n-1} \circ \cdots \circ f_2 \circ j_1 \circ f_1(x)\}$$

then for all $x \in X_1$:

$$^{\dagger}f_n(\cdot, \Delta_{n-1} \circ \cdots \circ \Delta_1(\delta)) \circ \cdots \circ {}^{\dagger}_{\Delta_2}f_2(\cdot, \Delta_1(\delta)) \circ {}^{\dagger}_{\Delta_1}f_1(x, \delta) \to j_n \circ f_n \circ j_{n-1} \circ \cdots \circ f_2 \circ j_1 \circ f_1(x)$$

as $\delta \to 0$. Moreover, this convergence is uniform on compact sets.

It is easy to construct examples that show that the implication in Theorem 1 does not reverse. The main reason for this appears to be that the result of the composition is sensitive to the choice of the $\overline{X}_i$s. It is possible to give a necessary criterion in terms of co-envelopes along the lines of [Neu22a, Theorem 2], but there remains a gap between the necessary and the sufficient condition. Our main open question is whether it is possible to find a more satisfactory necessary and sufficient criterion for convergence.

References

[Esc98] Martín Hötzel Escardó. Properly injective spaces and function spaces. *Topology and its Applications*, 89(1):75 – 120, 1998.

[Neu19] Eike Neumann. *Universal Envelopes of Discontinuous Functions*. PhD thesis, Aston University, 2019.

[Neu22a] Eike Neumann. On envelopes and backward approximations. In Ulrich Berger, Johanna N. Y. Franklin, Florin Manea, and Arno Pauly, editors, *Revolutions and Revelations in Computability - 18th Conference on Computability in Europe, CiE 2022, Swansea, UK, July 11-15, 2022, Proceedings*, volume 13359 of *Lecture Notes in Computer Science*, pages 220–233. Springer, 2022.

[Neu22b] Eike Neumann. Uniform envelopes. *Log. Methods Comput. Sci.*, 18(3), 2022.

[ZB04] Martin Ziegler and Vasco Brattka. Computability in linear algebra. *Theoretical Computer Science*, 326(1-3):187–211, 2004.