The equational theory of the Weihrauch lattice with multiplication and composition

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The talk would be based on the work presented in [6] and a continuation thereof.

The Weihrauch degrees $\mathfrak{W}$ come with a rich algebraic structure. Among the first operations on Weihrauch degrees studied in the literature [2, 1, 7] were the lattice operations $\sqcup, \sqcap$, the product $\times$ and the finite parallelization $(-)^*$. In order to better understand the structure of the Weihrauch degrees, we wanted to characterize its equational theory, i.e. identify which equations between terms over the signature $(\mathfrak{W}, \sqcap, \sqcup, \times, 1, (-)^*)$ are true for every instantiation of the variables by Weihrauch degrees. It was already observed in [3] that the equational theory of $(\mathfrak{W}, \sqcap, \sqcup)$ is the theory of distributive lattices, as $(\mathfrak{W}, \sqcap, \sqcup)$ is a distributive lattice itself, and every countable distributive lattice embeds into the Medvedev degrees and a fortiori $(\mathfrak{W}, \sqcap, \sqcup)$.

We started by investigating the equational theory of $(\mathfrak{W}, \sqcap, \times)$ in [6]. While we did not succeed in identifying a finite axiomatization for it (and in fact, conjecture that there might be none using only equations), we provided a combinatorial characterization in terms of reductions between finite graphs. This in turn allowed us to show that determining universal validity of equations in $(\mathfrak{W}, \sqcap, \times)$ is Σ_2^p-complete, and that this is Π_3^p-complete for the full signature $(\mathfrak{W}, \sqcup, \sqcap, \times, (-)^*, 0, 1)$. We also offered a sound axiomatization of the equational theory of $(\mathfrak{W}_\bullet, \sqcup, \sqcap, \times, (-)^*, 0, 1)$, where $\mathfrak{W}_\bullet$ is the subset of pointed Weihrauch degrees, that we conjecture to be complete. The aforementioned complexity results also hold for the pointed degrees.

We are not aware of axioms systems matching exactly those we provided for $(\mathfrak{W}_\bullet, \sqcap, \times)$ in the literature. Despite of this, there are tentative connections to be made between notions of correctness in substructural logics and our notion of reduction in terms of graphs. In particular, [4] has a strikingly similar notion of reducibility between graphs generalizing boolean formulas. Although our notion of reducibility is more complex, this seems to indicate related work in deep inference system might be of relevance in uncovering a complete axiomatization for the theories we consider.

In continuation of this work, we are currently investigating the equational theory associated to the composition operator $\star$, its iteration $(-)^\diamond$ and the lattice operations. This fragment is very similar to the language of *Kleene algebras* [5]. Kleene algebras admit a complete axiomatization, and that fact can be established by translating terms to automata and proving that inequality matches exactly language inclusion in the world of automata. A similar approach can be employed for the theory of $(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^\diamond, 0, 1)$. Since $\sqcup$ is not left-distributive over $\star$, we need to consider (variations of) the notion *simulations* between automata rather than language inclusions, and $\sqcap$ requires us to handle alternating automata rather than merely non-deterministic ones.

We propose an axiomatization of this equational theory and conjecture it to be complete using similar techniques as those deployed for Kleene algebras. We also conjecture it to be sound, which requires establishing that the entailment

$$x \star y \leq x \qquad \Rightarrow \qquad x \star y^\diamond \leq x$$

is valid in the Weihrauch degrees. For $x = y$, this was established in [8] via a non-trivial argument. We hope this can be adapted to prove the entailment above.

On the complexity-theoretic side of things, for $(\mathfrak{W}, \sqcup, \sqcap, \star, (-)^\diamond, 0, 1)$ and $(\mathfrak{W}, \sqcup, \sqcap, \star)$, deciding the validity of an equation seems PSPACE-complete in general. If we omit $\sqcap$, the problem is coNP-hard in general but polytime over pointed degrees.

References

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