Extracting solution operators for polynomial ODEs from proofs *

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The cAERN library [KPT24] is a formalization of exact real computation in the Coq proof assistant, based on the type-theoretical framework described in [KPT22b]. The library makes use of Coq's code extraction features to extract certified exact real computation programs on top of the Haskell library AERN. In previous work, we have shown that the approach allows to extract efficient programs for basic operations such as computing real and complex square roots [KPT22a]. We further extended the library to a theory of subsets, which can be used to generate accurate drawings of fractals up to any desired precision [KPT23].

Most recently, we consider the problem of solving initial value problems for polynomial ordinary differential equations. More precisely, we focus on equations of the form

$$\dot{y}(t) = p(y(t)) \ ; \ y(t_0) = y_0 \tag{1}$$

where $p : \mathbb{R}^d \to \mathbb{R}$ is a polynomial, $t_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}^d$ and $\dot{y}(t)$ denotes the derivative of the function $y(t)$ with respect to the time variable t.

The problem of computing the initial value problems for ordinary differential equation has been studies extensively in computable analysis and real number complexity theory (e.g. [Abe70, BGP12, Kaw10]) as well as in validated numerics (e.g. [Ned06]).

In this work, we are interested in computing the solution on some (preferably large) time interval $t \in [0, T]$ in the sense of computable analysis, i.e., we want to be able to approximate $y(t)$ for any $t \in [0, T]$ up to any desired output precision.

It is well known that the solution to (1) exists and is analytic in some neighborhood of 0. Further, by applying the usual chain rule for derivatives, we can compute the power series of the solution from the list of coefficients of the polynomial p. Algorithms based on this idea have also been studied for applications in exact real computation [BKM15, BKM16, KST18].

In this talk, we present an extension of cAERN to include solution operators for ODE solutions, and formal proofs of their correctness. To formulate the problem in our formalization, it is first necessary to extend our theory to statements about differentiation and some standard facts from classical analysis. Similar to [KST18] we then formalize a representation for analytic functions by their (infinite) power series. This functional representation can be used to approximate the value of the function on an interval up to any desired precision.

We show that we can compute such a representation for the solution of (1) valid on a small time interval around 0. Similar to single step methods in numerical analysis, we can then extend the solution to a larger interval by using the representation to compute a new initial value and iterating the algorithm.

From the proofs of correctness in Coq, we extract Haskell/AERN programs that solve arbitrary non-linear polynomial ODEs and outputs the solution trajectories.

Although the algorithm in [KST18] works for any dimension, our current formalization in Coq only supports one dimensional systems. In the talk we also present how the results can be extended to arbitrary dimension and more generic types of right-hand side functions.

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