Quotients of Weihrauch degrees

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We show that the following is a well-defined total operation on Weihrauch degrees:

$$a/b := \max_{\leq_W}\{c \mid b \times c \leq_W a\}$$

This answers a question from [1], where the idea of such an operation came up, the degree of D_2/LPO was investigated and the question was raised whether $\mathrm{SRT}^1_2/\mathrm{LPO}$ exists.

We observe that $0 <_W a/b$ iff $b \leq^*_W a$, that a/b is pointed iff $b \leq_W a$ and that $a^*/b \equiv_W a^* \times d_A$ where A is the (possibly empty) set of oracles witnessing $b \leq^*_W a^*$.

By definition we have that $a/a \equiv_W a$ iff $a \times a \equiv_W a$. Conversely, a degree a such that $a/a \equiv_W 1$ is as far away from being idempotent as possible. This property is satisfied for many non-idempotent degrees that have been studied in the literature:

Proposition 1. The following Weihrauch degrees all satisfy that $a/a \equiv_W 1$: C_k, LPO^k, CC_1, $\mathrm{TC}_{\mathbb{N}}$, Sort_2, RT^1_k

An intermediate behaviour is exhibited e.g. by the problem DS of finding an infinite descending sequence in an ill-founded linear order studied in [2]. Based on our recent results in [3], we can show that $\mathrm{C}_{\mathbb{N}} \leq_W \mathrm{DS}/\mathrm{DS} <_W \lim$. Another result from the literature we can recast in the language of quotients is $\mathrm{TC}_{\mathbb{N}^\mathbb{N}}/\mathrm{NON} \equiv_W \mathrm{C}_{\mathbb{N}^\mathbb{N}}$, which is [4, Proposition 8.2(3)].

There are some cases where a quotient of two well-studied Weihrauch degrees yields a third well-studied one, for example we have:

Proposition 2. $\mathrm{RT}^1_3/\mathrm{RT}^1_2 \equiv_W \mathrm{C}_2$

Proposition 3. $\mathrm{CC}_1/\mathrm{ACC}_{\mathbb{N}} \equiv_W \mathrm{C}^*_2$

The quotient of Weihrauch degrees extends the implication operator for the Medvedev degrees along the embedding $A \mapsto d_A : \mathfrak{M}^{\mathrm{op}} \to \mathfrak{W}$, where $d_A : A \to 1$ is the only map of its type. We have that $d_A/d_B = d_{B\to A}$, where $B \to A = \{np \mid \forall q \in B\ \Phi_n(\langle p,q\rangle) \downarrow\in A\}$.

We conclude with an observation on quotients of finite closed choice:

Proposition 4. $\mathrm{C}_n/\mathrm{C}_k \equiv_W \mathrm{C}_{\lfloor \frac{n}{k} \rfloor}$

References

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