# NEW DEFINITIONS IN THE THEORY OF TYPE 1 COMPUTABLE TOPOLOGICAL SPACES: EXTENDED ABSTRACT

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Abstract. We develop the Type 1 notion of "computable topological space". We show that a sufficiently general definition must rely on a notion of formal inclusion relation, as first used in this context by Dieter Spreen. We then consider several notions of effective bases. The first one, introduced by Nogina, is based on an effective version of the statement "a set $O$ is open if for any point in $O$, there is a basic set containing that point and contained in $O$". The second one, associated to Lacombe, is based on an effective version of "a set $O$ is open if it can be written as a union of basic open sets". An important, but little known, theorem of Moschovakis states that on "recursive Polish spaces", the two notions agree. Thanks to a modified and better behaved version of the Nogina notion of basis, we provide a new version of Moschovakis' Theorem, by relying solely on effective separability.

## 1. Introduction

There are several approaches to developing the notion of a "computable topological space". The presented paper [Rau23] deals with the Type 1 notion of computable topological spaces, that is to say the one which is based on the use of numberings. Here, we will even consider *non-surjective numberings* as a generalization of the usual concept of a numbering: a non-surjective numbering of a set $X$ is a partial map $\nu :\subseteq \mathbb{N} \to X$ which indicates that some points of $X$ (not necessarily all of them) admit finite descriptions.

The most successful approach to studying computability on topological spaces is probably the one used in computable analysis, and which is based on representations: a *representation* of a set $X$ is a partial surjection $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. In this case, we can consider that the notion of a "Type 2 computable topological space" is simply that of an admissibly represented space [Sch21]. Associated to a represented space $(X, \rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X)$ is a representation of the open sets of $X$ seen as continuous maps from $X$ to the Sierpiński space.

This representation of open sets via maps to the Sierpiński space can be seen as a special case of the notion of *intrinsic topology* used Realizability Theory. See for instance [Bau00], where the *standard dominance* is introduced: it is a realizability generalization of the Sierpiński space with its usual representation.

Thus, in the study of represented spaces, one always restricts their attention to the intrinsic topology of the considered sets.

However, a crucial fact in the Type 2 theory of effectivity is that this "restriction" is actually not one: in practice, *when one wants to study a topological space $X$, it is always possible to find an admissible representation $\rho$ of $X$, so that the intrinsic topology of $\rho$ be exactly the topology of $X$ that we want to study.* (The topological spaces where this is possible were characterized by Schröder as the $\text{qcb}_0$ spaces [Sch03].)

In the study of Type 1 computability, the corresponding phenomenon does not occur.

The intrinsic topology of a numbered set is called the *Ershov topology*: it is the topology of semi-decidable sets.

And it is not possible to study Type 1 computable topologies by restricting our attention to Ershov topologies, because Type 1 Ershov topologies cannot be "calibrated" to match any desired abstract topology. The most famous example of this phenomenon is Friedberg's theorem of existence of a semi-decidable set of computable real numbers which is not open in the usual topology of $\mathbb{R}_c$. This example was explained by Hoyrup and Rojas in [HR16] in terms of Kolmogorov complexity.

Thus there is more work to be done in order to arrive to a robust Type 1 definition of a computable topological space.

## 2. Contents

We briefly describe the contents of [Rau23].

2.1. **Formal inclusion relations.** Let $X$ be a set, and $\beta :\subseteq \mathbb{N} \to \mathcal{P}(X)$ a non-surjective numbering of a set of subsets of $X$. A *formal inclusion relation* for $\beta$ is a relation $\mathring{\subseteq}$ on $\text{dom}(\beta)$ such that:

- $\mathring{\subseteq}$ is reflexive and transitive;
- $\forall n, m \in \text{dom}(\beta),\ n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m)$.

This is a special case of the more general notion of strong inclusion relation used by Spreen in [Spr98], where reflexivity is not asked.

2.2. **Computable topological spaces.** The following is the proposed general definition of computable topological space.

**Definition 2.1.** A *Type 1 computable topological space* is a quintuple $(X, \nu, \mathcal{T}, \tau, \mathring{\subseteq})$ where $X$ is a set, $\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on $X$, $\nu :\subseteq \mathbb{N} \to X$ is a non-surjective numbering of $X$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}$ is a non-surjective numbering of $\mathcal{T}$, $\mathring{\subseteq}$ is a formal inclusion relation for $\tau$, and such that:

1. The image of $\tau$ generates the topology $\mathcal{T}$;
2. The empty set and $X$ both belong to the image of $\tau$;
3. The open sets in the image of $\tau$ are uniformly semi-decidable;
4. The operations of taking computable unions and finite intersections are computable, i.e.:
   (a) The function $\bigcup : \mathcal{T}^{\mathbb{N}} \to \mathcal{T}$ which maps a computable sequence of open sets $(A_n)_{n\in\mathbb{N}}$ to its union is computable, and it can be computed by a function that is expanding and increasing for $\mathring{\subseteq}$;
   (b) The function $\bigcap : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ which maps a pair of open sets to their intersection is computable, and it can be computed by a function increasing for $\mathring{\subseteq}$.

2.3. **Different notions of bases.** One of the interesting aspects of studying bases for computable topologies is that one is very quickly confronted to classically equivalent statements which yield non-equivalent effective definitions. Indeed, there are two equivalent formulations of the classical definition of a basis for a topology on a set $X$. A basis is a set $\mathcal{B}$ of subsets of $X$ such that:

- Every element of $X$ belongs to an element of $\mathcal{B}$;
- For any two elements $B_1$ and $B_2$ of $\mathcal{B}$, and for any $x$ in $B_1 \cap B_2$, there is an element $B_3$ in $\mathcal{B}$ containing $x$ and such that $B_3$ is a subset of $B_1 \cap B_2$.
- A subset $O$ of $X$ is called *open* if for any $x$ in $O$ there is $B$ in $\mathcal{B}$ such that $x \in B$ and $B \subseteq O$.

Or, a second approach:

- The union of elements of $\mathcal{B}$ gives $X$:
$$\bigcup_{B\in\mathcal{B}} B = X;$$
- The intersection of two elements of the basis can be written as a union of elements of this basis: for any $B_1$ and $B_2$ in $\mathcal{B}$, there is a subset $\mathcal{C}$ of $\mathcal{B}$ such that
$$B_1 \cap B_2 = \bigcup_{B\in\mathcal{C}} B.$$
- A subset $O$ of $X$ is called *open* if it can be written as a union of basic sets.

Both of these approaches are classically equivalent, but they differ in terms of computational content. We call the corresponding notions of computable bases *Nogina bases* (because of [Nog66]) and *Lacombe bases* (following the article [Lac57], Lachlan [Lac64] and Moschovakis [Mos64] were the first ones to use the term "Lacombe set").

A theorem of Moschovakis [Mos64] states that on countable sets that are "recursively Polish", the two notions of bases agree. Thanks to a new notion of computable basis, based on Nogina's but that also uses formal inclusions, we provide a new version of Moschovakis' Theorem, that takes effective separability as sole hypothesis.

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