BANACH-MAZUR COMPUTABLE ANALYSIS ON THE SPACE OF MARKED GROUPS

EMMANUEL RAUZY

When studying computability in group theory, one distinguishes between *global* and *local* problems. A local problem is a problem about the elements of a group, a global problem is a problem whose input data consists in whole groups. The description most commonly used to study global problems is that of a *finite presentation of a group*. A remarkable fact is that equality is semi-decidable for groups given by finite presentations. In computable analysis, a semi-decidable equality is seen as a form of "effective discreteness": this explains why topological methods are not useful in the study of decision problems for finitely presented groups.

However, recent developments due to Groves, Manning and Wilton [GW09, GMW12] have made it clear that it is necessary to study other types of descriptions of groups, and in particular that one should study *groups given by the solution to their word problem*. We will see that computable analysis then becomes relevant.

Groups given by their solution to the word problem. Fix a countable group G. A finite set of elements $S = (s_1, s_2, ...s_k)$ in G *generates* G if every element in G can be obtained as a product of the elements $\{s_1, ...s_k\}$ and of their inverses. Thus every element of G can be seen as a word of $(S \cup S^{-1})^*$, but several words can define the same group element. The *word problem* in (G, S) is exactly the problem of deciding which words in $(S \cup S^{-1})^*$ define the same group element.

If we identify $(S \cup S^{-1})^*$ and $\mathbb{N}$ via a computable bijection ψ, we see that the word problem of (G, S) can be seen as an element of the Cantor space: it is encoded by the sequence $(u_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ given by $u_{\langle p,q\rangle} = 1$ if and only if $\psi(p)$ and $\psi(q)$ are two words of $(S \cup S^{-1})^*$ that define the same element in G. Thus the word problem in (G, S) can be seen as an element of the Cantor space.

Lemma 1. *A group is uniquely determined by its word problem.*

Thanks to this lemma, we can see that the map $\rho_{WP} :\subseteq \{0, 1\}^{\mathbb{N}} \to \mathcal{G}$ which maps a sequence $(u_n)_{n\in\mathbb{N}}$ as above to the unique group it defines is a representation of the set of finitely generates groups. Note that not all sequences in $\{0, 1\}^{\mathbb{N}}$ define a group.

We can now study computability on the represented space $(\mathcal{G}, \rho_{WP})$. However, as we will see, we will also need older notions of computability.

Different approaches to computability. In computable analysis, settling on a single formalism took years. There were many different proposals for what should be the correct notion of a "computable function of a real variable". Thus it is only natural that, when the problem of choosing a notion of computability on infinite data arose in group theory, in a context where people were unaware that similar problems had been investigated in analysis, several answers were proposed.

In fact, when Groves, Manning and Wilton decided to study groups given by the solution to their word problem in [GMW12], they settled for a form of *Banach-Mazur computability*. Recall that a function is Banach-Mazur computable when it maps computable sequences (of computable points) to computable sequences. In other words, if it is Markov computable on all computable sequences.

The main problem of Banach-Mazur computability is that in practice, positive results, that establish that something is indeed computable, always provide more than Banach-Mazur computability. Thus to prove that a function is Banach-Mazur computable, one proves that it is computable with respect to the natural associated representations, and then *weakens* that result to Banach-Mazur computability.

On the other hand, the use of Banach-Mazur computability (and of Markov computability) will remain relevant when establishing *negative* results, that prove that something is not computable. Indeed, these negative results are stronger than those stated in terms of Type 2 computability, using representations.

There are some theorems which allow to automatically obtain Markov undecidability results from topological results, the so called *continuity theorems*, as the Rice-Shapiro Theorem, the Kreisel-Lacombe-Schoenfield-Ceitin Theorem [KLS57, Cei67], and Spreen's unification and generalization of these results [Spr98].

However, $\mathcal{G}$ is a c.e. closed subset of the Cantor space which is not co-c.e. closed. And thus the represented space $(\mathcal{G}, \rho_{WP})$ is not a computable metric space, because it is not computably separable. This prevents us from applying continuity results.

Higman's Embedding Theorem and Banach-Mazur computability. In computational group theory, there is a strong interest in finitely presented groups. In particular, we are very much interested in the construction of finitely presented groups inside of which some problems are undecidable. The theorem that requires us to investigate Banach-Mazur computability is the following one:

Theorem 2 (Higman [Hig61], Clapham [Cla67])**.** *A finitely generated group is a subgroup of a finitely presented group with decidable word problem if and only if it has decidable word problem.*

A corollary to this theorem is:

Corollary 3 (R.)**.** *A group property P is not Banach-Mazur semi-decidable if and only if there exists a finitely presentable group G with solvable word problem such that the problem of determining if a finitely generated subgroup of G has P is not semi-decidable.*

And thus it is only when we establish that a property is not Banach-Mazur semi-decidable that we can conclude that there is one single finitely presented group that witnesses for this undecidability.

Descriptive set theory. We then classify group properties in $\mathcal{G}$. We are interested in three different measures of complexity: the topological complexity of the property, in terms of descriptive set theory, the complexity in the effective Borel hierarchy that comes from the representation ρ_{WP}, and finally, restricting our attention to computable points, the complexity in terms of Markov computability and of Banach-Mazur computability.

Many properties are precisely classified. However, there are examples of properties:

- Whose topological classification is not known (Burnside groups of exponent 5: groups such that $\forall g, g^5 = 1$ are known to form a closed set, but it may also be open);
- Which are known to be open or closed, but for which the Type 2 effective classification is unknown (finitely presented simple groups define an open set which is not c.e. open if the well-known conjecture of Boone-Higman holds);
- Which are classified both topologically and in Type 2, but not in terms of Markov/Banach-Mazur computability (the closure of the set of finite groups is closed, c.e. closed, but not co-c.e. closed, but we can only conjecture that it is not co-semi-decidable in terms of Banach-Mazur computability).

Finally, note that the closure of the set of finite groups is compact but not computably compact.

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