Principal topological spaces

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The weak ultrafilter axiom WUF postulates the existence of an ultrafilter $\mathcal{U}$ on the subsets of $\mathbb{N}$ that is *free*, meaning $\bigcap(\mathcal{U}) = \emptyset$ (cf. [2]). It is well-known that ZFC validates WUF. By contrast, in Shelah's model of set theory, $\mathsf{ZF} + \mathsf{DC} + \mathsf{BP}$, the negation of WUF is true (cf. [2, 29.37]). As a consequence, every prime filter $\mathcal{F}$ on $\mathsf{O}(\mathbb{N})$ is *principal*, which means that there is some $n \in \mathbb{N}$ such that $\mathcal{F} = \{M \subseteq \mathbb{N} \,|\, n \in M\}$.

We investigate the class of qcb_0-spaces which have the latter property. Qcb_0-spaces play a big role in Type Two Theory of Effectivity (TTE) [4]. They form the class of topological spaces which can be handled by TTE, cf. [3]. In the sequel we are working in $\mathsf{ZF} + \mathsf{DC}$, where DC stands for the Axiom of Dependent Choice.

Principal spaces: Definition and Examples

Let X be a topological space. Remember that a *prime filter* $\mathcal{F}$ on the lattice $\mathsf{O}(X)$ is a non-empty family of open subsets of X that does not contain $\emptyset$ and is upwards-closed, closed under forming finite intersections, and *prime* in the sense that $U \cup V \in \mathcal{F}$ implies $U \in \mathcal{F}$ or $V \in \mathcal{F}$ for all $U, V \in \mathsf{O}(X)$.

We define X to be a *principal space*, if every prime filter $\mathcal{F}$ on $\mathsf{O}(X)$ is equal to the open neighbourbood filter $\{U \text{ open} \,|\, x \in U\}$ of some unique point $x \in X$. Filters generated by a point are usually called "principal". Clearly, any principal space is T_0 and sober.

It is easy to see that all finite T_0-spaces are principal. Moreover, the sobrification of $\mathbb{N}$ equipped with the co-finite topology is principal. By contrast, the Scott domain $\mathcal{P}(\mathbb{N})$ is not principal. In ZFC a qcb_0-space is principal iff it is sober and Noetherian. Moreover, in ZFC all infinite Hausdorff spaces are not principal. The latter follows from:

Proposition 1 *If there exists an infinite principal Hausdorff space, then* $\neg\mathsf{WUF}$ *holds.*

On the positive side, $\neg\mathsf{WUF}$ yields a big supply of principal spaces. Important examples are:

Theorem 2 *In* $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WUF}$ *every functionally Hausdorff qcb-space is principal.*

Thus the question whether or not the Euclidean space $\mathbb{R}$ is principal depends on the axiomatic setting, and it is unanswerable in $\mathsf{ZF} + \mathsf{DC}$. Functionally Hausdorff qcb-spaces have nice closure properties and encompass all separable metrisable spaces and many spaces used in Functional Analysis (cf. [3]).

Proposition 3 *The category of functionally Hausdorff qcb-spaces is cartesian-closed and has all countable limits and all countable co-products.*

The Axiom of Determinacy AD used in Game Theory is known to imply the Baire Property Axiom BP and thus $\neg\mathsf{WUF}$. Moreover, $\mathsf{ZF} + \mathsf{DC} + \neg\mathsf{WUF}$ is equiconsistent with ZFC.

Applications of principal spaces

Principality has some extraordinary consequences. For example, it implies the following automatic continuity property.

Proposition 4 *Let Y be a principal space and X be a topological space. Let $h\colon \mathsf{O}(Y) \to \mathsf{O}(X)$ be a function that preserves binary intersection and binary union. Then:*

(1) *h is Scott-continuous.*

(2) *h preserves arbitrary unions.*

Bounded lattice homomorphisms are in bijective correspondence with continuous functions.

Proposition 5 *Let Y be a principal space and X be a topological space. Let $h\colon \mathsf{O}(Y) \to \mathsf{O}(X)$ be a bounded lattice homomorphism (i.e. h preserves $\bot \mathrel{\widehat{=}} \emptyset, \cap, \cup, \top$).*
Then there is a function $f\colon X \to Y$ satisfying $f^{-1}[V] = h(V)$ for all open subsets $V \subseteq Y$.

Recently prime ideals on commutative rings got some interest in Computable Analysis [1].

Proposition 6 *Let X be a principal T_6-space. Then for every prime ideal $\mathcal{I}$ on the commutative ring $C(X, \mathbb{R})$ there is a unique $z \in X$ such that z is a zero of all functions in $\mathcal{I}$; moreover, there is a function $g \in \mathcal{I}$ such that $g^{-1}\{0\} = \{z\}$.*

By contrast, in ZFC there is a prime ideal $\mathcal{I}$ on $C(\mathbb{R}, \mathbb{R})$ such that $\bigcap\{f^{-1}\{0\} \mid f \in \mathcal{I}\}$ is empty. The question arises for which spaces all prime ideals are even generated by some point z of X. Up to now the answer is only known for the case of (certain) zero-dimensional spaces.

Proposition 7 *Let X be a principal zero-dimensional hereditarily Lindelöf space. Then for every prime ideal $\mathcal{I}$ on $C(X, \mathbb{R})$ there is some $z \in X$ such that $\mathcal{I} = \{f \in C(X, \mathbb{R}) \mid f(z) = 0\}$.*

As an open problem we ask whether Proposition 7 can be extended to all principal T_6-spaces.

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References

[1] M. de Brecht: *A note on the closed prime spectrums of coPolish commutative rings.* Proceedings of CCA 2022.

[2] E. Schechter: *Handbook of Analysis and Its Foundations.* Academic Press, San Diego (1997).

[3] M. Schröder: *Admissibly Represented Spaces and QCB-Spaces.* In: Handbook of Computability and Complexity in Analysis, Springer, 305–346 (2021). Available as arXiv: 2004.09450

[4] K. Weihrauch: *Computable Analysis.* Springer, Berlin (2000).