Chains and antichains in the Weihrauch degrees

Manlio Valenti[1][0000−0003−0351−3058]

Swansea University manlio.valenti@swansea.ac.uk

Despite the growing amount of results on the properties of Weihrauch reducibility, most of the efforts tend to concentrate on the characterization of the degree of specific problems whose interest comes from reverse mathematics or classical analysis. However, many of the structural properties of the Weihrauch degrees are unknown and did not receive enough attention in the literature. We will discuss some results about the existence of chains and antichains in the Weihrauch degrees.

It is known that every countable set of Weihrauch degrees has an upper bound. This can be witnessed by taking the countable coproduct of representatives of each degree. This is not a degree-theoretic operation though, and in fact, the supremum of a countable set of degrees exists only in the trivial case where the supremum is already in the set. In particular, no chain of degrees of order type ω has a supremum.

It is not hard to show that there is a chain of order type ω_1 in the Weihrauch degrees without an upper bound (which implies that there is no general way to extend the countable coproduct to the uncountable case). In fact, we provide a characterization of the sequences $(f_\alpha)_{\alpha<\kappa}$ that have an upper bound.

Theorem 1. *Let κ be a cardinal with $\operatorname{cof}(\kappa) > \omega$, and let $(f_\alpha)_{\alpha<\kappa}$ be a chain of order type κ. Let $I_p^E := \{\alpha \in E : p \in \operatorname{dom}(f_\alpha)\}$. The following are equivalent:*

1. *$(f_\alpha)_{\alpha<\kappa}$ has an upper bound in $\mathcal{W}$;*
2. *there is a cofinal $E \subset \kappa$ s.t. for every $p \in \bigcup_{\alpha\in E} \operatorname{dom}(f_\alpha)$, $\bigcap_{\alpha\in I_p^E} f_\alpha(p) \neq \emptyset$.*

This is also related to the problem of determining the cofinality of the Weihrauch degrees. It is known that the existence of a cofinal chain in the Turing degrees is equivalent to CH. The same equivalence holds by replacing the Turing degrees with the Medvedev degrees. However, for the Weihrauch degrees, we can prove the following in ZFC.

Theorem 2. *There are no cofinal chains in $\mathcal{W}$.*

Finally, we turn our attention to the extendibility of antichains. Dzhafarov, Lerman, Patey, and Solomon have shown that for every countable family $\{f_n\}_{n\in\mathbb{N}}$ of non-trivial problems there is g such that for every n, $g \mid_{\mathrm{W}} f_n$. This result cannot be extended to continuum-sized families.

Proposition 1. *There is a continuum-sized family $\{f_p\}_{p\in\mathbb{N}^\mathbb{N}}$ such that for every non-trivial g there is $p \in \mathbb{N}^\mathbb{N}$ such that $f_p \leq_{\mathrm{W}} g$.*

This family cannot be refined to an uncountable antichain. Indeed, we obtain the following sufficient condition for extendibility.

Theorem 3. *Let* $\mathcal{M}_0$ *denote the Medvedev degrees without the top element. Let* $\{f_\alpha\}_{\alpha<\mathfrak{c}}$ *be an antichain in* $\mathcal{W}$ *and let* $D_\alpha := \operatorname{dom}(f_\alpha)$*. If the set* $\mathcal{B} := \{D_\beta : (\forall x \in D_\beta)(f_\beta(x) \le_{\mathrm{M}} \{x\})\}$ *is not cofinal in* $\mathcal{M}_0$ *then* $\{f_\alpha\}_{\alpha<\mathfrak{c}}$ *is not maximal.*

All these results showcase how, despite the close interplay between Medvedev and Weihrauch reducibility, the two lattices have a very different structure.

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