Computable Type and Weihrauch Complexity

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A general question

What are the relations between the computability of a set and its topological properties?

In other words, how can the topological properties of a set affect its computability?

Computability of a set

- A subset K of the plane is semicomputable if there is an algorithm which enumerates progressively balls that do not intersect K.

- K is computable if it is semicomputable and there is an algorithm which enumerates progressively balls that intersect K.

An natural question

When are the two previous notions of computability equivalent?

Outline

Semicomputable/computable sets

We endow $\mathbb{R}^n$ with the topology generated by rational balls $(B_i)_{i\in\mathbb{N}}$.

Definition

A compact set K in $\mathbb{R}^n$ is

1. Semicomputable if the set
$$\{i \in \mathbb{N} : K \cap \overline{B}_i = \emptyset\}$$
is c.e.
2. Computable if it is semicomputable and the set
$$\{i \in \mathbb{N} : K \cap B_i \neq \emptyset\}$$
is c.e.

The line segment

Fact

One can build a segment $[a, 1]$ which is semicomputable but not computable.

Proof.

Create a real number $a = \sum_{n \in A} 2^{-n-1}$ where $A \subseteq \mathbb{N}$ is the halting set (a non-computable c.e. set). The segment $[a, 1]$ is semicomputable but not computable. □

What about the circle? Is it possible to have a similar proof?

Previous results

Some semicomputable sets are actually computable:

- The circle, and more generally n-dimensional spheres [Miller 2002].
- Closed manifolds [Iljazovic et al. 2013].
- A characterization of finite simplicial complexes satisfying this property [A., Hoyrup 2022].
- ...

An interesting question

For which sets one has the equivalence

$$\text{Semicomputable} \Longleftrightarrow \text{computable ?}$$

Computable type

Definition

The Hilbert cube is the complete metrizable space $Q = [0, 1]^{\mathbb{N}}$.

Fact

Every computable metric space embeds effectively into the Hilbert cube.

Definition (Iljazovic)

A compact metrizable space X has computable type if every semicomputable copy of X in the Hilbert cube is computable.

Strong computable type

Definition (A., Hoyrup)

A compact space X has strong computable type if for every oracle O, every copy of X which is semicomputable relative to O must be computable relative to O.

Remark

Spaces from the literature like spheres and manifolds have strong computable type.

A sufficient condition

Recall that the circle in $\mathbb{R}^2$ has strong computable type.
Observe that it has a hole but no proper subset of it has a hole,

so the circle is minimal satisfying the property of "having a hole".

A sufficient condition

We obtain the following sufficient condition to have strong computable type using topological invariants.

Theorem (A., Hoyrup)

If X is $\mathcal{P}$-minimal for some Σ^0_2 topological invariant $\mathcal{P}$, then X has strong computable type.

We proved that this condition is not necessary.

Extension to pairs

Note that the notion of computable type extends to pairs.

For instance, the pair consisting of the line segment and its two endpoints has computable type (Iljazovic 2020).

To simplify, in this presentation we focus on single sets.

Our goal

We want to study the relation between computable type and Weihrauch complexity.

Outline

Weihrauch reducibility

Definition

Let $f, g :\subseteq X \rightrightarrows Y$ be multi-valued functions. We say that f is a strengthening of g and we write $f \sqsubseteq g$ if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and for every $x \in \mathrm{dom}(g)$, $f(x) \subseteq g(x)$.

A represented space (X, δ) is a set X together with a surjective partial function $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

A problem is a partial multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces X and Y.

Given represented spaces (X, δ_X), (Y, δ_Y), a problem $f :\subseteq X \rightrightarrows Y$ and a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. We say that F is a realizer of f and we write $F \vdash f$ if $\delta_Y F \sqsubseteq f \delta_X$.

Weihrauch reducibility

Definition

Let f and g be problems. f is Weihrauch reducible to g, denoted $f \leq_W g$, if there exist computable functions $K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $H :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that for every $G \vdash g$,

$$H \circ (\mathrm{id}, G \circ K) \vdash f$$

where

$$\begin{aligned}(\mathrm{id}, G \circ K) : \mathbb{N}^\mathbb{N} &\to \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \\ p &\mapsto (p, G(K(p)))\end{aligned}$$

Strong Weihrauch reducibility

Definition

Let f and g be problems. f is strongly Weihrauch reducible to g, denoted $f \leq_{sW} g$, if there exist computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for every $G \vdash g$, $H \circ G \circ K \vdash f$.

- Strong Weihrauch reducibility implies Weihrauch reducibility.
- $\leq_W$ and $\leq_{sW}$ are preorders, i.e. they are reflexive and transitive. The corresponding equivalences are denoted by $\equiv_W$ and $\equiv_{sW}$ respectively.
- Strong Weihrauch reducibility relative to some oracle is usually denoted by $\leq^t_{sW}$.

Vietoris topology

Definition

Let $\mathcal{K}(Q)$ be the hyperspace of Q, i.e. the space of non-empty compact subsets of Q.
It can be equipped with:

1. The upper Vietoris topology $\tau_{up\mathcal{V}}$ generated by the sets of the form
$$\{K \in \mathcal{K}(Q) : K \subseteq U\},$$
where U ranges over the open subsets of Q.
2. The Vietoris topology $\tau_{\mathcal{V}}$ generated by $\tau_{up\mathcal{V}}$ and the sets of the form
$$\{K \in \mathcal{K}(Q) : K \cap U \neq \emptyset\},$$
where U ranges over the open subsets of Q.

Vietoris topology and Hausdorff distance

Definition

Let $A, B \subseteq Q$ be two non-empty compact sets, the Hausdorff distance between A and B is defined by:

$$d_H(A,B) = \max\left(\max_{a\in A}\min_{b\in B} d_Q(a,b), \max_{b\in B}\min_{a\in A} d_Q(a,b)\right).$$

The space $(\mathcal{K}(Q), \tau_{\mathcal{V}})$ is a computable Polish space, by taking the Hausdorff metric.

Vietoris topology and computability notions

For a compact set $K \subseteq Q$,

$$K \text{ is semicomputable} \iff K \text{ is a computable element of } (\mathcal{K}(Q), \tau_{up\mathcal{V}})$$
$$K \text{ is computable} \iff K \text{ is a computable element of } (\mathcal{K}(Q), \tau_{\mathcal{V}})$$

A natural question

- Given a compact space X which has strong computable type, is it possible to have a single effective procedure that takes any copy Y given in the topology $\tau_{up\mathcal{V}}$ and computes Y in the topology $\tau_{\mathcal{V}}$?
- Uniformity is only possible when X is a singleton.
- What is the degree of non-uniformity of this problem, using Weihrauch degrees? It was studied by [Pauly 2021] in the case of the circle embedded in $\mathbb{R}^2$.

SCT_X and Closed choice

Definition

For a compact space X, let SCT_X be the function taking a copy Y of X in $\tau_{up\mathcal{V}}$ and outputting Y in $\tau_{\mathcal{V}}$.

Definition

Closed choice over $\mathbb{N}$ is the problem $\mathrm{C}_{\mathbb{N}}$ of finding an element in a non-empty set A of natural numbers, given any enumeration of the complement of A.

Outline

SCT_X and Closed choice

Theorem (A., Hoyrup)

Let X be a compact space which is not a singleton. One has $\mathsf{C}_{\mathbb{N}} \leq_{sW}^{t} \mathsf{SCT}_X$. If X has a semicomputable copy then $\mathsf{C}_{\mathbb{N}} \leq_{sW} \mathsf{SCT}_X$.

This result holds for every compact space (which may or may not have strong computable type).

SCT_X and Closed choice

Proof.

- Instead of $C_{\mathbb{N}}$, we use the strongly Weihrauch equivalent problem Max which sends a non-empty finite subset of $\mathbb{N}$ to its maximal element.
- We prove the result when X has a semicomputable copy, the general case is obtained by relativizing the argument to an oracle which semicomputes some copy.

□

SCT_X and Closed choice

Proof.

- Given a non-empty finite set $E \subseteq \mathbb{N}$, let $m = \max E$.
- We produce a semicomputable copy X_E of X such that $2^{-m-1} < \mathrm{diam}(X_E) < 2^{-m}$.
- Given an access to X_E in the topology $\tau_{\mathcal{V}}$, one can compute its diameter so one can compute m.
- Hence, Max is strongly Weihrauch reducible to SCT_X.

□

Computable type and Closed choice

Fact (A., Hoyrup)

If X is $\mathcal{P}$-minimal for some Σ_2^0 topological invariant $\mathcal{P}$, then X has strong computable type.

Theorem (A., Hoyrup)

Let $\mathcal{P}$ be a Σ_2^0 invariant in $\tau_{up\mathcal{V}}$. If a compact space X is $\mathcal{P}$-minimal then $\mathsf{SCT}_X \leq_W \mathsf{C}_{\mathbb{N}}$.

A strong Weihrauch reduction is impossible.

Computable type and Closed choice

Proof.

- As $\mathcal{K}(Q)$ is a metric space, $\mathcal{P}$ is of the form $\mathcal{P} = \bigcup_n \mathcal{P}_n$ where $\mathcal{P}_n$ are uniformly Π^0_1-sets in $\tau_{up\mathcal{V}}$.
- Given a copy Y, let $E = \{n : Y \in \mathcal{P}_n\}$. From Y in $\tau_{up\mathcal{V}}$ one can enumerate the complement of E.
- Given any $n \in E$, one can compute Y by using the fact that Y is $\mathcal{P}_n$-minimal:
  - Given the $\tau_{up\mathcal{V}}$ neighborhoods of X, we need to enumerate the rational balls U intersecting X.
  - U intersects X iff $X \setminus U$ is a proper subset of X
  - As X is $\mathcal{P}_n$-minimal, it is equivalent to $X \setminus U \notin \mathcal{P}_n$.
  - We can compute $X \setminus U$ in the topology $\tau_{up\mathcal{V}}$ so we can semi-decide whether $X \setminus U \notin \mathcal{P}_n$, i.e. whether U intersects X.

Computable type and Closed choice

Corollary

Let X be a compact space which is not a singleton and which has a semicomputable copy. Let $\mathcal{P}$ be a Σ^0_2 invariant in $\tau_{up\mathcal{V}}$. If X is $\mathcal{P}$-minimal then $\mathsf{SCT}_X \equiv_W \mathsf{C}_{\mathbb{N}}$.

An open question

Question

Is it ***always*** *true that if X has strong computable type, then $\mathrm{SCT}_X \leq_W \mathrm{C}_{\mathbb{N}}$?*

- **Publications** (Amir & Hoyrup)
  - The Surjection Property and Computable Type, Topology and its Applications, 2024.
  - Strong Computable Type, Computability, 2023.
  - Comparing Computability in Two Topologies, Journal of Symbolic Logic (JSL), 2023.
  - Computability of Finite Simplicial Complexes, ICALP 2022.
- **Computability of Topological Spaces**, PhD Thesis, HAL, 2023.
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Thank you!