Computability of irreducible continua

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16.7.2024.

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Set $S \subseteq \mathbb{N}^k$ is recursively enumerable if it is an image of some recursive function from $\mathbb{N}$ to $\mathbb{N}^k$.

Proposition

If $S, T \subseteq \mathbb{N}^k$ are recursively enumerable, then $S \cap T$ and $S \cup T$ are recursively enumerable.

Computable metric space

Function $f : \mathbb{N}^k \to \mathbb{R}$ is recursive if there exists recursive function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that

$$|f(x) - F(x,n)| < 2^{-n}, \ \forall x \in \mathbb{N}^k \ \forall n \in \mathbb{N}.$$

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Computable metric space (X, d, α) is metric space (X, d) together with sequence $\alpha : \mathbb{N} \to X$ whose image is dense in X and such that function $\mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_J)$ is recursive.

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Example

If $\alpha : \mathbb{N} \to \mathbb{Q}$ is recursive surjection, then $\alpha(\mathbb{N})$ is dense in $\mathbb{R}$, and function $(i, j) \mapsto |\alpha_i - \alpha_j|$ is recursive. That means that $\mathbb{R}$ with standard metric and sequence α is computable metric space.

Computable point

We say that $x \in \mathbb{R}$ is a computable number if there exists a recursive function $f : \mathbb{N} \to \mathbb{Q}$ such that

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Let (X, d, α) be a computable metric space. We say that $x \in X$ is a computable point if there exists a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that

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Sequence $x : \mathbb{N} \to X$ is computable if there exists recursive function $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$d(x_n, \alpha_{g(n,k)}) < 2^{-k}, \ \forall n, k \in \mathbb{N}.$$

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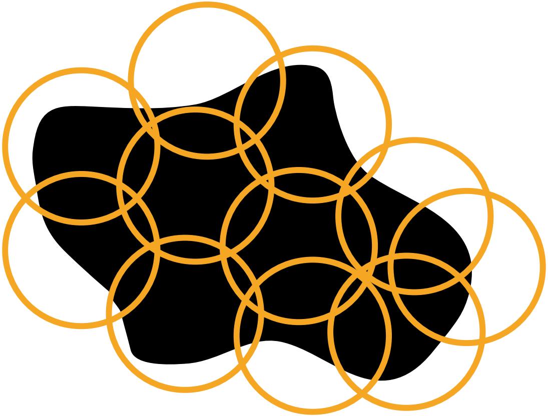
Compact set $S \subseteq X$ is computable if there exists recursive function $f : \mathbb{N} \to \mathbb{N}$ such that

$$S \approx_{2^{-k}} \alpha([f(k)]), \ \forall k \in \mathbb{N}.$$

Computable set



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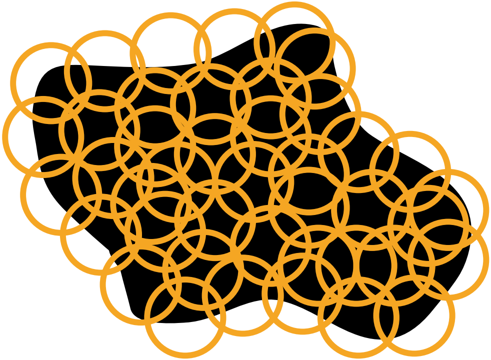

Rational open sets

For $i \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$, we say that $B(\alpha_i, r)$ is a rational open ball in (X, d, α). Finite unions of rational open balls are called rational open sets.

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If $q : \mathbb{N} \to \mathbb{Q}$ is recursive function, and $q(\mathbb{N}) = \mathbb{Q} \cap \langle 0, \infty \rangle$, we can recursively enumerate the set of all rational open balls using q and α.
We denote that sequence of rational open balls with $(I_i)_{i \in \mathbb{N}}$.

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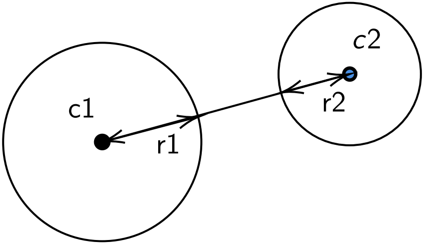
Then

$$(J_j)_{j \in \mathbb{N}}, \;\; J_j = \bigcup_{i \in [j]} I_i$$

is a sequence of all rational open sets.

We define relation $\diamond$:
For $I_i = B(c_i, r_i)$ and $I_j = B(c_j, r_j)$ we define $I_i \diamond I_j$ iff $d(c_i, c_j) > r_i + r_j$.



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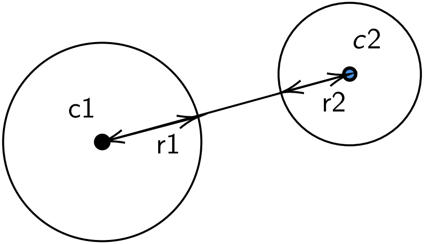
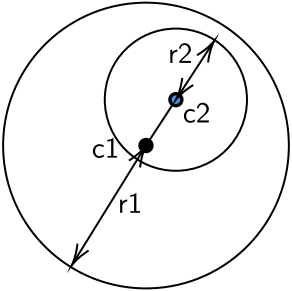


$$I_i \diamond I_j \implies I_i \cap I_j = \emptyset$$

For J_u and J_v we define $J_u \diamond J_v$ iff $\forall i \in [u]\ \forall j \in [v]\ I_i \diamond I_j$.
$J_u \diamond J_v \implies J_u \cap J_v = \emptyset$
Set $\{(u, v) \in \mathbb{N}^2 \mid J_u \diamond J_v\}$ is recursively enumerable.

We define relation $\subseteq_F$:
For $I_i = B(c_i, r_i)$ and $I_j = B(c_j, r_j)$ we define $I_i \subseteq_F I_j$ iff $d(c_i, c_j) + r_i < r_j$.



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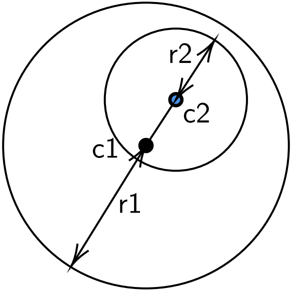

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If $(x)_{n\in\mathbb{N}}$ is a computable sequence, then set $\overline{\{x(n) \mid n \in \mathbb{N}\}}$ is computably enumerable.
If $S \subseteq X$ is computably enumerable and complete, then there exists a computable sequence $(x)_{n\in\mathbb{N}}$ such that $S = \overline{\{x(n) \mid n \in \mathbb{N}\}}$.

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$$S \text{ computable} \Longrightarrow S \text{ semicomputable}$$
$$S \text{ semicomputable} \not\Longrightarrow S \text{ computable}$$

semicomputability $\Longrightarrow$ computability

Semicomputability does imply computability if we have some additional topological assumptions on S. For example, if S is homeomorphic to the sphere S^n.

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Arc is a topological space homeomorphic to the set $[0,1]$.
Let L be an arc, and $f : [0,1] \to L$ homeomorphism. Then $f(0)$ and $f(1)$ are called the endpoints of L.

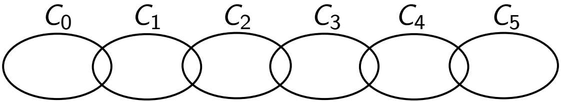
Theorem

Let (X, d, α) be a computable metric space, and $S \subseteq X$ semicomputable. If S is a semicomputable arc with computable endpoints, then S is computable.

Chainable continuum

Let X be a set. Finite sequence $C_0, \ldots, C_n$ of subsets of X is called a chain in X if

$$C_i \cap C_j \neq \emptyset \Longleftrightarrow |i - j| \leq 1.$$

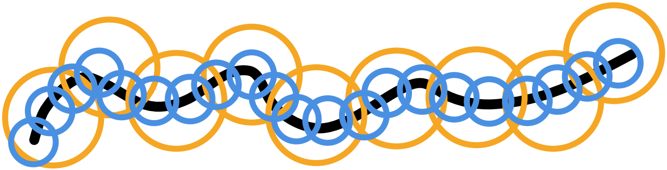


Chain in metric space is said to be an open $\varepsilon-$chain if all links C_i are open sets, and diam $C_i < \varepsilon$ for $i = 0, \ldots, n$.

Continuum is a compact connected metric space.

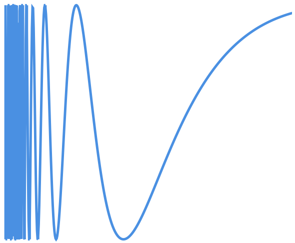
Chainable continuum

Let (X, d) be continuum, and $a, b \in X$. Continuum X is said to be chainable from a to b if for every $\varepsilon > 0$ there exists open $\varepsilon-$chain $C_0, \ldots, C_n$ such that $a \in C_0$, $b \in C_n$ and $X = \bigcup_{i=0}^{n} C_i$.



Chainable continuum

$$\{0\} \times [-1, 1] \cup \{(x, \sin \tfrac{1}{x}) \mid 0 < x \leq a\}$$



Chainable continuum

Theorem

Let (X, d, α) be a computable metric space, and $S \subseteq X$ semicomputable. If S is a continuum chainable from a to b, where a and b are computable points, then S is computable.

Important step in the proof is showing that continuum X chainable from a to b has the following property:

for every $x \in X \setminus \{a, b\}$ and every $\varepsilon > 0$ there exist compact sets $F, G \subseteq X$ such that $F \cap G = \emptyset$, $a \in F$, $b \in G$ and $X = F \cup B(x, \varepsilon) \cup G$. $(*)$

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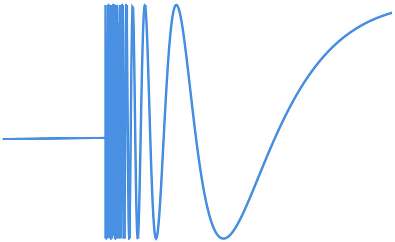
We have shown that continuum X has that property if and only if it is irreducible between a and b, i.e. there are no proper subcontinua of X that contain both a and b.

Irreducible continuum



Irreducible continuum

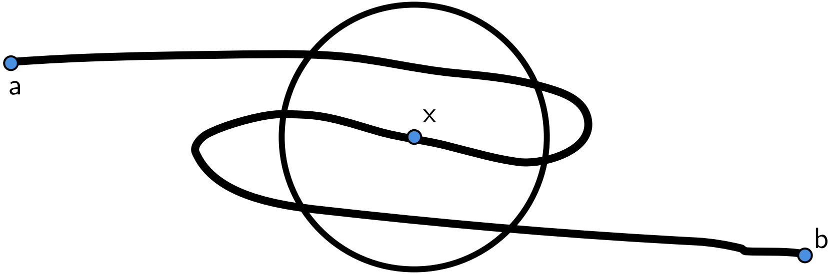
Theorem

Let (X, d, α) be a computable metric space, and $S \subseteq X$ semicomputable. If S is a continuum irreducible from a to b, where a and b are computable points, then S is computable.

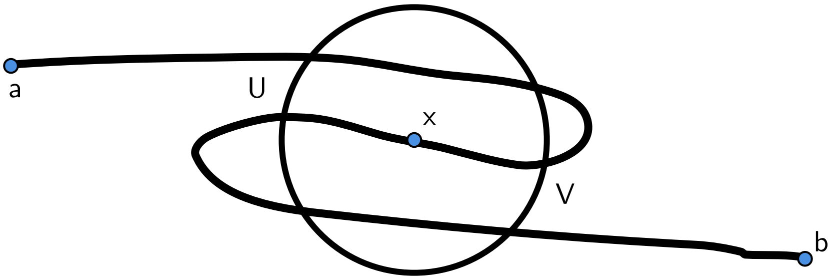
Proof

If S is irreducible between a and b, then S has property $(*)$:

let $x \in S \setminus \{a, b\}$, and $r > 0$ such that $a, b \notin B(x, r)$.
From irreducibility of S it follows that a and b belong to different connected components of $S \setminus B(x, r)$.



It is a known result that in that case, there exists (U, V) a separation of $S \setminus B(x, r)$, such that $a \in U$ and $b \in V$.
Since U, V are closed in S, they are also compact.
We have disjoint compact sets U, V such that $a \in U$, $b \in V$ and $S = U \cup B(x, r) \cup V$. Therefore, S has property $(*)$.



Proof

If S has property $(*)$, then S is irreducible between a and b:

If we assume that S is not irreducible between a and b, then there exists a proper subcontinuum $a, b \in K \subset S$. Set $S \setminus K$ is nonempty and open in S, so there exist a point $x \in S \setminus \{a, b\}$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \cap K = \emptyset$.
Points a and b are in the same connected component of $S \setminus B(x, \varepsilon)$. So there can not exist two disjoint compacts F, G in $S \setminus B(x, \varepsilon)$ such that $a \in F$ and $b \in G$. That is in contradiction with property $(*)$.

Proof

Set S is computable iff it is semicomputable and computably enumerable.
Assumption: S is semicomputable continuum irreducible between computable points a and b.
We need to show that S is computably enumerable, i.e. that set $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is recursively enumerable.
We have $I_i \cap S \neq \emptyset$ iff there exist $u, v \in \mathbb{N}$ such that $J_u \diamond J_v$, $S \subseteq J_u \cup I_i \cup J_v$, $a \in J_u$ and $b \in J_v$.

If $I_i \cap S \neq \emptyset$, then there exist $x \in S \setminus \{a, b\}$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq I_i \cap S$.
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There exist $u, v \in \mathbb{N}$ such that $F \subseteq J_u$, $G \subseteq J_v$ and $J_u \diamond J_v$.
We have shown that if $I_i \cap S \neq \emptyset$, then there exist $u, v \in \mathbb{N}$ such that $J_u \diamond J_v$, $S \subseteq J_u \cup I_i \cup J_v$, $a \in J_u$ and $b \in J_v$.

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Converse also holds.

If $I_i \cap S \neq \emptyset$, then there exist $x \in S \setminus \{a, b\}$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq I_i \cap S$.
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Converse also holds.

Set $\{(i, u, v) \in \mathbb{N}^3 \mid J_u \diamond J_v, S \subseteq J_u \cup I_i \cup J_v, a \in J_u, b \in J_v\}$ is recursively enumerable, so $\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$ is also recursively enumerable.

□

Computable points

If S is computable set, then it contains computable point. Moreover, computable points are dense in S.

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Does semicomputable set necessarily contain computable points?

No.

Theorem

Let (X, d, α) be a computable metric space and $S \subseteq X$ a semicomputable arc with endpoints a and b. Then for every $\varepsilon > 0$ there exist computable points $\hat{a}, \hat{b} \in S$, $d(a, \hat{a}) < \varepsilon$, $d(b, \hat{b}) < \varepsilon$ such that the subarc of S with endpoints $\hat{a}$ and $\hat{b}$ is a computable set in (X, d, α).

Theorem

Let (X, d, α) be a computable metric space and $S \subseteq X$ semicomputable, decomposable, chainable continuum. Then for every $\varepsilon > 0$ there exist computable points $\hat{a}, \hat{b} \in S$ and a computable subcontinuum $\hat{S}$ of S such that $\hat{S} \approx_\varepsilon S$ and $\hat{S}$ is chainable from $\hat{a}$ to $\hat{b}$.

Our result

Let (X, d, α) be a computable metric space and $S \subseteq X$ semicomputable continuum irreducible beetween a and b. If there exist proper subcontinua $K_1, K_2, K_3 \subseteq S$ such that $S = K_1 \cup K_2 \cup K_3$ and $S \neq K_i \cup K_j$, for $i, j \in 1, 2, 3$, then S contains an open set in which computable points are dense.

Proof

WLOG $a \in K_1$ and $b \in K_2$. Since $S \setminus (K_1 \cup K_2)$ is nonepty and open in S, there exists $i_0 \in \mathbb{N}$ such that $\emptyset \neq I_{i_0} \cap S \subseteq S \setminus (K_1 \cup K_2)$.

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$$a \notin K_2 \cup K_3 \implies \exists \tilde{a} \in \mathbb{N} \text{ such that } a \notin J_{\tilde{a}} \text{ and } K_2 \cup K_3 \subseteq J_{\tilde{a}}$$

$$b \notin K_1 \cup K_3 \implies \exists \tilde{b} \in \mathbb{N} \text{ such that } b \notin J_{\tilde{b}} \text{ and } K_1 \cup K_3 \subseteq J_{\tilde{b}}$$

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$$b \notin K_1 \cup K_3 \implies \exists \tilde{b} \in \mathbb{N} \text{ such that } b \notin J_{\tilde{b}} \text{ and } K_1 \cup K_3 \subseteq J_{\tilde{b}}$$

We claim that $\overline{I_{i_0} \cap S}$ is computably enumerable, i.e. that set $\{i \in \mathbb{N} \mid I_i \cap \overline{I_{i_0} \cap S} \neq \emptyset\}$ is recursively enumerable.

Assume $I_i \cap \overline{I_{i_0} \cap S} \neq \emptyset$, then there exist $x \in S$, $r > 0$ such that $B(x, r) \subseteq I_i \cap I_{i_0}$ and $B(x, r) \subseteq S \setminus (K_1 \cup K_2)$. Continuum S has property $(*)$, so there exist disjoint compacts $F, G \subseteq S$ such that $a \in F$, $b \in G$ and $S = F \cup B(x, r) \cup G$.

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It follows that $K_1 \cup K_2 \subseteq F \cup G$, and since K_1, K_2 are connected, $K_1 \subseteq F$ and $K_2 \subseteq G$.

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It follows that $K_1 \cup K_2 \subseteq F \cup G$, and since K_1, K_2 are connected, $K_1 \subseteq F$ and $K_2 \subseteq G$.

$$K_1 \cap G = \emptyset \Longrightarrow G \subseteq S \setminus K_1 \Longrightarrow G \subseteq K_2 \cup K_3 \subseteq J_{\tilde{a}}$$
$$K_2 \cap F = \emptyset \Longrightarrow F \subseteq S \setminus K_2 \Longrightarrow F \subseteq K_1 \cup K_3 \subseteq J_{\tilde{b}}$$

There exist $u, v, w \in \mathbb{N}$ such that $J_u \diamond J_v$, $F \subseteq J_u \subseteq_F J_{\tilde{b}}$ and $G \subseteq J_V \subseteq_F J_{\tilde{a}}$, and $B(x, r) \subseteq J_w \subseteq_F I_i$.

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We define

$$\Omega = \{(i, u, v, w) \in \mathbb{N}^4 \mid J_u \diamond J_v, J_u \subseteq_F J_{\tilde{b}}, J_v \subseteq_F J_{\tilde{a}}, J_w \subseteq I_i, \\ S \subseteq J_u \cup J_v \cup J_w\}.$$

We have $I_i \cap \overline{I_{i_0} \cap S} \neq \emptyset$ iff there exist $u, v, w \in \mathbb{N}$ such that $(i, u, v, w) \in \Omega$. Since Ω is recursively enumerable, then $\{i \in \mathbb{N} \mid I_i \cap \overline{I_{i_0} \cap S} \neq \emptyset\}$ is also recursively enumerable.

□

Thank you for your attention!