Computability of One-Point Metric Bases
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Outline

Metric Basis - Motivation

Example

Consider a segment in $\mathbb{R}$ with endpoints x_0 and x_1.

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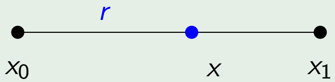

An interesting property of x_0: any point x of the segment is *uniquely* determined by its *distance* r from an endpoint x_0.

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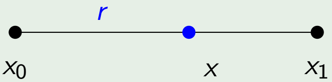

An interesting property of x_0: any point x of the segment is *uniquely* determined by its *distance* r from an endpoint x_0.

Note

It can be shown that the same property holds for the endpoint x_1*.*

Metric Basis - Definition

Definition (Melter and Tomescu [2])

Suppose (X, d) *is a metric space, let* $S \subseteq X$ *be a non-empty set such that for all* $x, y \in X$ *the following implication holds:*

$$\textit{if } d(s,x) = d(s,y) \textit{ for each } s \in S, \ \textit{ then } x = y.$$

Then we say that S *is a* ***metric basis*** *for* (X, d).

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Definition

Let (X, d) *be a metric space. We say that a point* $x_0 \in X$ *is a one-point metric basis for* (X, d) *if* $\{x_0\}$ *is a metric basis for* (X, d).

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Definition

Let (X, d) be a metric space. We say that a point $x_0 \in X$ is a one-point metric basis for (X, d) if $\{x_0\}$ is a metric basis for (X, d).

Note

Any subset of X containing a metric basis for (X, d) is a metric basis for (X, d).

Main question

Question

Let (X, d, α) be a computable metric space and $x_0 \in X$ a metric basis for (X, d). Under which conditions is x_0 computable in (X, d, α)?

Note

We study this question for some computable metric spaces that are:

- *compact and connected*
- *compact and disconnected with finitely many components*
- *non-compact and connected*

Some know computability results for arcs

A connection between the computability of an arc and the computability of its endpoints has been very well studied.

- Miller has shown in [3] that there exists a computable arc in $\mathbb{R}^2$ with noncomputable endpoints.
- However, a computable arc in $\mathbb{R}$ has to be of the form $[a, b]$, where a and b are computable real numbers.
- On the other hand, it is known that if endpoints of a semicomputable arc are computable, then the arc is computable ([1, 4]).

Computable Metric Spaces - Definitions

Definition

A triple (X, d, α) is a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is a sequence with a dense image in X such that the function $\mathbb{N}^2 \to \mathbb{R}$

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable. We call the points $\alpha_0, \alpha_1, \ldots$ **rational points** or **special points**.

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Definition

A *point* $x \in X$ is **computable** in (X, d, α) if there is a computable function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$d(x, \alpha_{f(k)}) < 2^{-k}$$

for all $k \in \mathbb{N}$.

Computable Metric Spaces - Definitions (contd.)

Definition

- A set I is a **rational ball** if $I = B(\lambda, \rho)$ where λ is a rational point and $\rho \in \mathbb{Q}^+$.
- We denote by (I_k) and $(\widehat{I_k})$ some fixed effective enumerations of open and closed rational balls respectively.

Definition

- A finite union of open rational balls is called **rational open set**.
- Denote by (J_j) some fixed effective enumeration of rational open sets.

Definition

Let (X, d, α) be a computable metric space. Let $S \subseteq X$ be closed.

1. S is **computably enumerable** in (X, d, α) i.e. the set

$$\{i \in \mathbb{N} : S \cap I_i \neq \emptyset\}$$

   is recursively enumerable.

2. S is a **co-computably enumerable set** in (X, d, α) if there exists a c.e. $\Omega \subseteq \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \Omega} I_i$$

3. S is **semi-computable** if S is compact and

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

   is recursively enumerable.

4. S is **computable** if S is semi-computable and computably enumerable.

Effective Compactness

Definition

A metric space (X, d) is said to be **effectively compact** if (X, d) is compact and there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X = B(\alpha_0, 2^{-k}) \cup \cdots \cup B(\alpha_{f(k)}, 2^{-k})$$

for each $k \in \mathbb{N}$.

Result for connected compact spaces

Theorem

Assume that (X, d, α) is an effectively compact computable metric space such that the space (X, d) is connected. If $x_0 \in X$ is metric basis for (X, d), then x_0 is a computable point in (X, d, α).

Proof (sketch)

Idea

Prove that $\{x_0\}$ *is co-computably enumerable: find a r.e. set* $\Omega \subseteq \mathbb{N}$ *such that*

$$X \setminus \{x_0\} = \bigcup_{i \in \Omega} I_i.$$

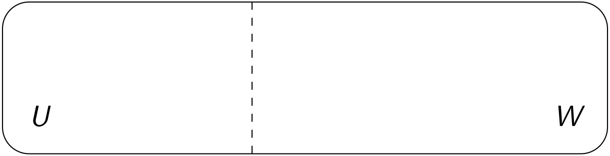
Lemma

Lemma

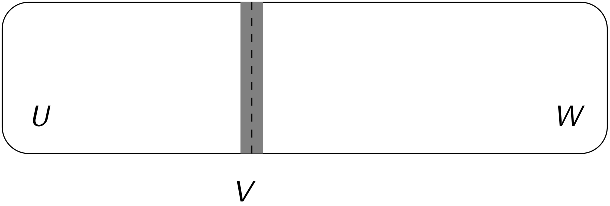
Let (X, d) be a connected metric space. Let $\varepsilon > 0$ and U, V, W open sets in (X, d) such that $U \cup V \cup W = X$, $U \cap W = \emptyset$ and $\mathrm{diam}\ V < \varepsilon$.

If there exist $x_1 \in U$, $x_2 \in V$, $x_3 \in W$ such that $d(x_1, x_2) > 2\varepsilon$ and $d(x_2, x_3) > 2\varepsilon$, then V does not contain a point which is a metric basis of (X, d).

Lemma



Lemma



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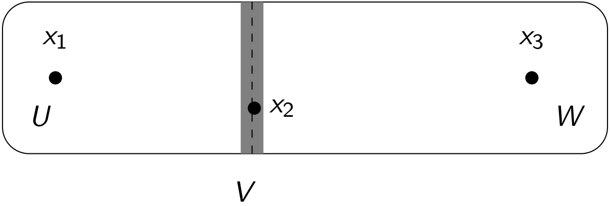

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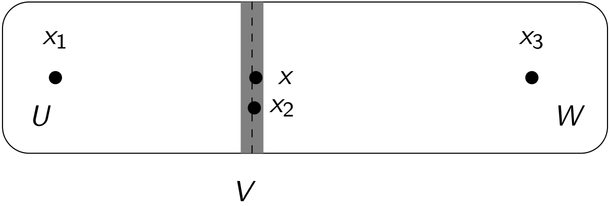

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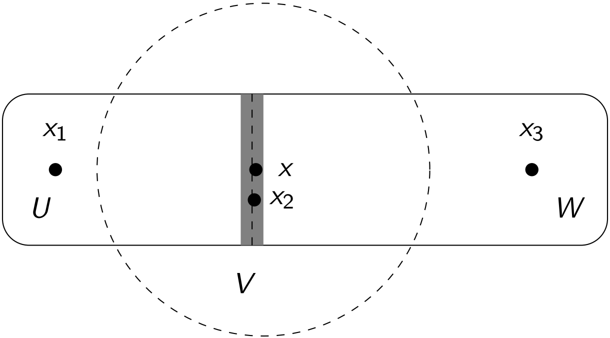

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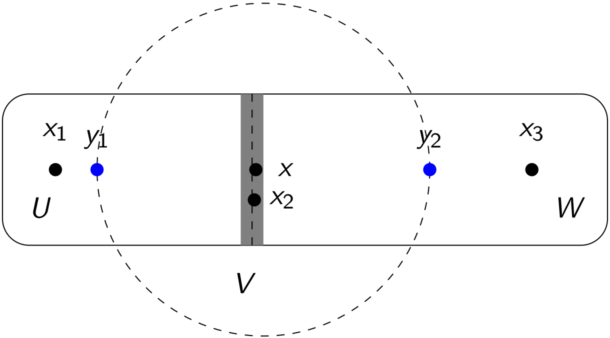

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Proposition

If (X, d) is a compact and connected metric space which has a metric basis then (X, d) is an arc.

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Proposition

Let (X, d) be an arc. If x_0 is a metric basis for (X, d) then x_0 is an endpoint.

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Note

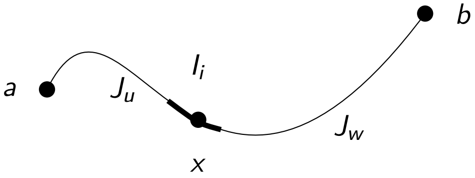
Hence, (X, d, α) is an effectively computable arc with endpoints a and b and the metric basis $x_0 \in \{a, b\}$. In the following, we continue the proof for $x_0 = a$. Proof for $x_0 = b$ is analogous.

Proof (sketch)

Proposition

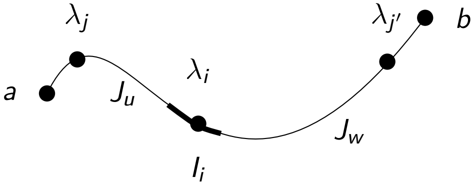
There is c.e. set $\Omega \subseteq \mathbb{N}$ such that the condition $i \in \Omega$ implies the conditions of the Lemma when applied to an effectively compact arc.

To prove $X \setminus \{a\} \subseteq \bigcup_{i\in\Omega}$: For a given $x \notin \{a, b\}$, and some $\varepsilon > 0$, we can always find the sets J_u, J_w and I_i such that $a \in J_u$, $b \in J_w$ and $x \in I_i$ and that satisfy certain covering conditions similar to the ones from the Lemma, but chosen to imply $i \in \Omega$.



Proof (sketch)

To prove $\bigcup_{i\in\Omega} I_i \subseteq X \setminus \{a\}$: $i \in \Omega$ implies conditions of the Lemma, which in turn implies that I_i does not contain a metric basis point, hence, it does not intersect a.



Proof (sketch)

- We conclude that $\{a\}$ is *co-computably enumerable* in (X, d, α).
- From effective compactness of (X, d, α) it is not hard to show that $\{a\}$ co-computably enumerable implies that a is a computable point in (X, d, α).

Example

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The effective compactness assumption cannot be removed as the following example shows.

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Example

Let $\gamma > 0$ be left-computable but not computable real. Consider computable metric space constructed from a segment in $\mathbb{R}$ with endpoints 0 and γ.

Note that this space is not effectively compact. Now γ is a metric basis of this space, but γ is not computable.

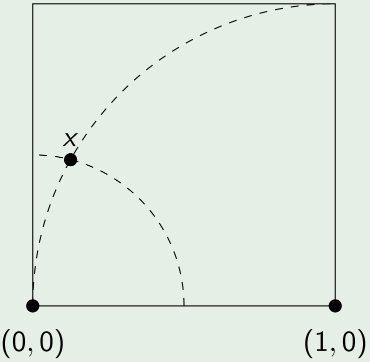
Example

Note

Further, the claim of the Theorem does not hold for metric bases which contain more than one point.

Example

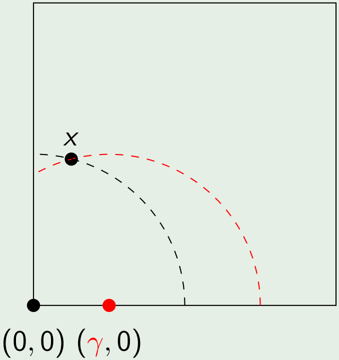
Consider I, the unit square $[0,1]^2$ with euclidean metric.



Here, $\{(0,0),(1,0)\}$ is a two-point metric basis for I and both of its points are computable.

Example

On the other hand, for $\gamma \in \langle 0, 1 \rangle$ uncomputable, we also have the following.



Here, $\{(0,0), (\gamma, 0)\}$ is a two-point metric basis for I, which contains a non-computable point.

Further results

Theorem

Assume that (X, d, α) is effectively compact computable metric space such that the space (X, d) has finitely many connected components. If $x_0 \in X$ is metric basis for (X, d), then x_0 is a computable point in (X, d, α).

Further results

Theorem

Assume that (X, d, α) is effectively compact computable metric space such that the space (X, d) has finitely many connected components. If $x_0 \in X$ is metric basis for (X, d), then x_0 is a computable point in (X, d, α).

The proof uses the following facts.

Proposition

Let (X, d) be compact and disconnected with finitely many components. If (X, d) has a one-point metric basis then (X, d) is an union of disjoint arcs.

Proposition

(X, d, α) is effectively compact if and only if X is a computable set in (X, d, α).

Definition

Let (X, d, α) be a computable metric space. We say that (X, d, α) has **effective covering property** if

$$\{(i,j) \in \mathbb{N}^2 \mid \widehat{I}_i \subseteq J_j\}$$

is recursively enumerable.

Definition

A metric space (X, d) is a **topological ray** if (X, d) is homeomorphic to $[0, +\infty\rangle$.

Proposition

Let (X, d) be a connected metric space which is not compact. If (X, d) has a metric basis then (X, d) is a topological ray.

Result for topological ray

Theorem

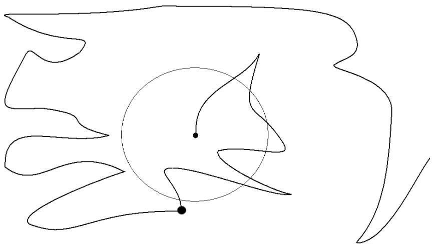
Let (X, d, α) be a computable metric space which is a topological ray and that has the effective covering property and compact closed balls. If $x_0 \in X$ is a metric basis for (X, d) then x_0 is a computable point in (X, d, α).

Proof (sketch)

Idea

Similar proof as for the arc: given the metric basis $x_0 \in X$ *a new r.e. set* $\Omega \subseteq \mathbb{N}$ *is constructed such that* $X \setminus \{x_0\} = \bigcup_{i \in \Omega} I_i$ *taking the following into account:*

- *if* x_0 *is a metric basis then* x_0 *is an endpoint*
- *a semi-computability notion for closed subsets that are not compact*
- *the "unboundedness" of the topological ray*



Summary

Theorem

Let (X, d, α) be a computable metric space. Let $x_0 \in X$ be a metric basis for (X, d).

1. *If (X, d, α) is effectively compact and (X, d) is connected then x_0 is computable in (X, d, α).*
2. *If (X, d, α) is effectively compact and (X, d) is disconnected with finitely many connected components then x_0 is computable in (X, d, α).*
3. *If (X, d, α) is connected and has compact closed balls and the effective covering property and (X, d) is not compact then x_0 is computable in (X, d, α).*

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Thank you for your attention!