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On the computational complexity of unfriendly partitions

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CCA (Swansea)

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Notations, definitions

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- *countable*,
- *undirected*,
- and *no self-loops*.

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Informally, for a partition to be unfriendly, we require that any vertex has at most as many neighbors in the opposite partition than in its own one.

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Remark: Milner, Shelah also showed that any graph has an *unfriendly 3-partition*.

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(Aharoni, Milner, Prikry)

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In this talk, we are interested in understanding

how hard is to find an unfriendly partition of a graph?

Weihrauch reducibility

Problems

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Definition

A represented space $\mathbf{X}$ is a pair $(X, \delta_{\mathbf{X}})$ where $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.
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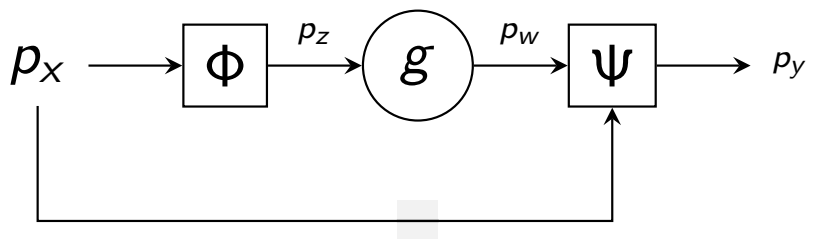
$\mathrm{UP}_{\mathcal{F}}$

Input: $G \in \mathcal{F}$.
Output: an unfriendly partition of G (an element of $2^{\mathbb{N}}$).

Let f and g be problems on represented spaces. Then f is *Weihrauch reducible* to a problem g ($f \leq_{\mathrm{W}} g$), if there are computable maps $\Phi, \Psi : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ s.t.

- for every name p_x for some $x \in dom(f)$, $\Phi(p_x) = p_z$, where p_z is a name for some $z \in dom(g)$ and,
- for every name p_w for some $w \in g(z)$, $\Psi(p_x \oplus p_w) = p_y$ where p_y is a name for $y \in f(x)$.



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It is easy to notice that $\mathrm{UP}_{\mathcal{F}}$ is *parallelizable*, i.e., $\mathrm{UP}_{\mathcal{F}} \equiv_{\mathrm{W}} \widehat{\mathrm{UP}_{\mathcal{F}}}$.

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LPO

Input: $p \in 2^{\mathbb{N}}$.
Output: 0 if $p = 0^{\mathbb{N}}$, 1 otherwise.

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Sketch of $\mathrm{UP}_{\mathcal{F}} \leq_{\mathrm{W}} \widehat{\mathrm{LPO}}$
Gura, Hirst and Mummert showed that $\widehat{\mathrm{LPO}}$ is enough to compute all disconnected components of a graph. Then, we can computably find an unfriendly partition of a finite graph.

Sketch $\widehat{\mathsf{LPO}} \leq_{\mathrm{W}} \mathsf{UP}_{\mathcal{F}}$ (Shafer)

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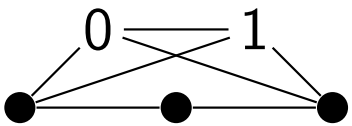

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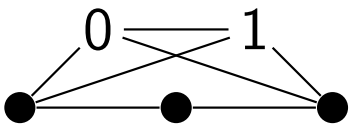

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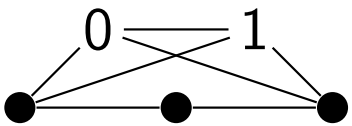

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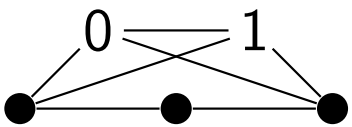

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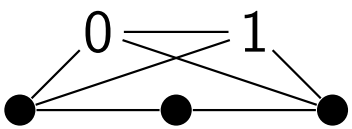

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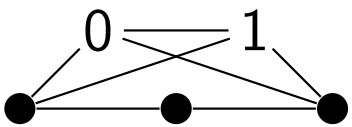

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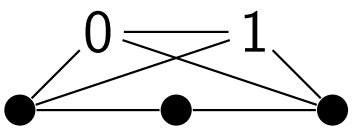

forces 0 and 1 to be in <u>the same</u> partition;

Hence, given any u.p. of G, we have that 0 and 1 are in different partitions iff $p = 0^{\mathbb{N}}$ iff $\text{LPO}(p) = 0$.

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Theorem

$\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{UP}_{\mathrm{Forests}} \equiv_{\mathrm{W}} \mathsf{UP}_{\mathrm{Bipartite}}$.

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Let T be an ill-founded[1] binary tree.

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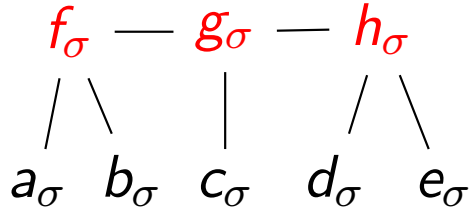

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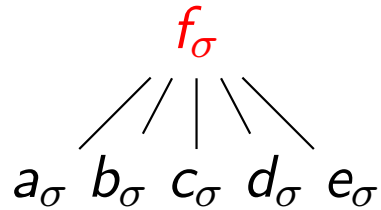

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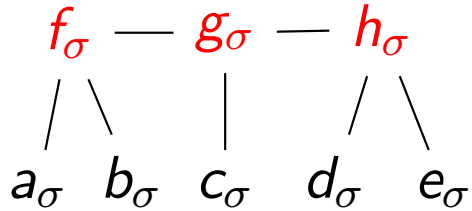

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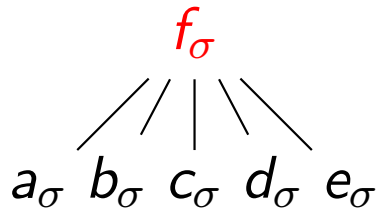

Verification Any u.p. of

- the lhs forces $a_\sigma - e_\sigma$ to be in different partitions,
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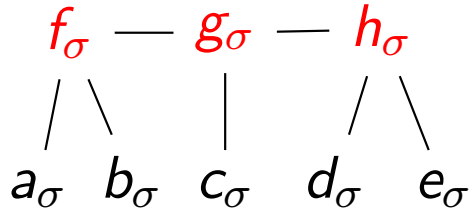

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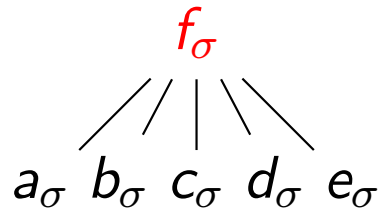

Verification Any u.p. of

- the lhs forces $a_\sigma - e_\sigma$ to be in different partitions,
- the rhs forces $a_\sigma - e_\sigma$ to be in the same partition.

Given an u.p. of G we compute $p \in [T]$ letting $p(n+1) = 0$ if the u.p. does not have $a_{p[n]} - e_{p[n]}$ in the same partition and 1 otherwise.

[1] being ill-founded is a Π^0_1 condition and hence well-founded is Σ^0_1

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Theorem

KL $\equiv_{\mathrm{W}}$ $\mathsf{UP}_{\mathcal{LF}}$.

"Stratifying" $\mathrm{UP}_{\mathcal{F}}$

Notice that:

- not having a subdivision of a 3-clique is the same as bipartite;
- we proved that $\mathrm{UP}_{\mathcal{F}_4} \equiv_{\mathrm{W}} \mathrm{lim}$ for $\mathcal{F}_4$ being the class of graphs with only finite components and no subdivision of the infinite 4-clique.

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Question: are there other natural classes of graphs that can be characterized in terms of well-known problems?

Above arithmetic

These (and other) results that we have can also be rephrased as

"There is a computable graph G s.t. any u.p. of G computes

- $\emptyset'$
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Something more?

Theorem

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Before giving a (rough) description of the graph mentioned in the theorem, we describe some of its parts: namely the *NOT* and *NAND* gates.

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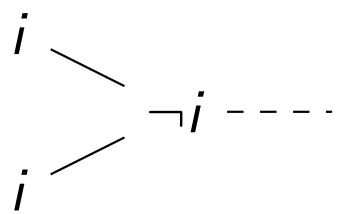

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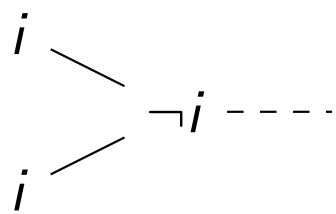
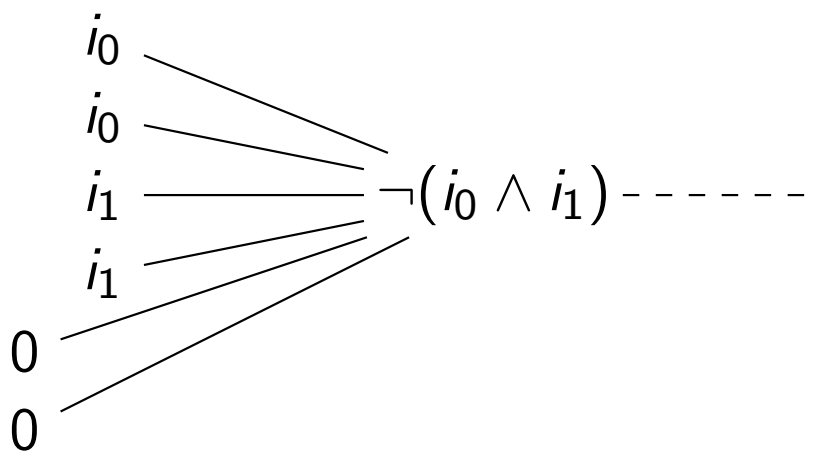


The gate does not "work" if the i's are in different partitions.

NAND gate

If all input vertices (labelled by i_0, i_1) are in the same partition (say 1) and the vertices labelled 0 are in the opposite one then the vertex labelled $\neg(i_0 \wedge i_1)$ must be in partition 0 (provided $\neg(i_0 \wedge i_1)$ has at most seven neighbors).



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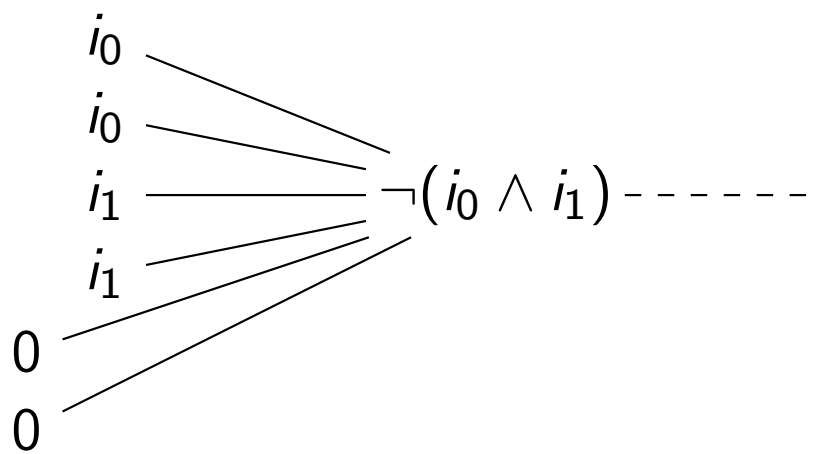

Differently from the NOT gate, such gate needs the "gadget" inputs (labelled 0).

Combinations of NAND and NOT

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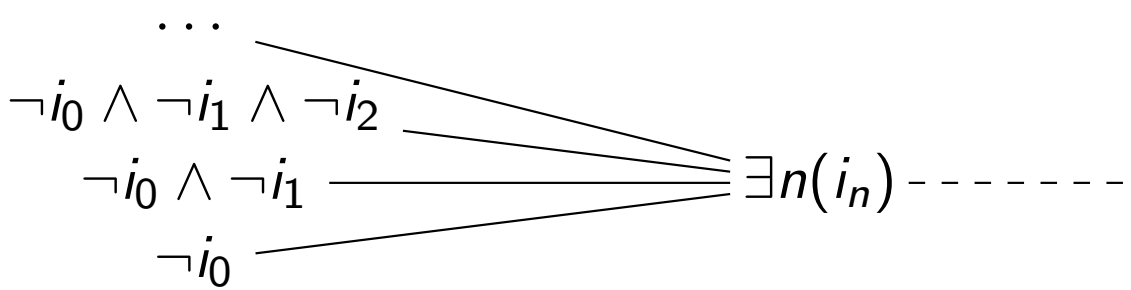
This may in turn be used to form a gate for $\exists$: Indeed:

- if i_n fails for all n, then $\neg i_0 \wedge \cdots \wedge \neg i_n$ holds for all n,
- if i_n holds for some n, then $\neg i_0 \wedge \cdots \wedge \neg i_n$ fails for cofinitely many n.



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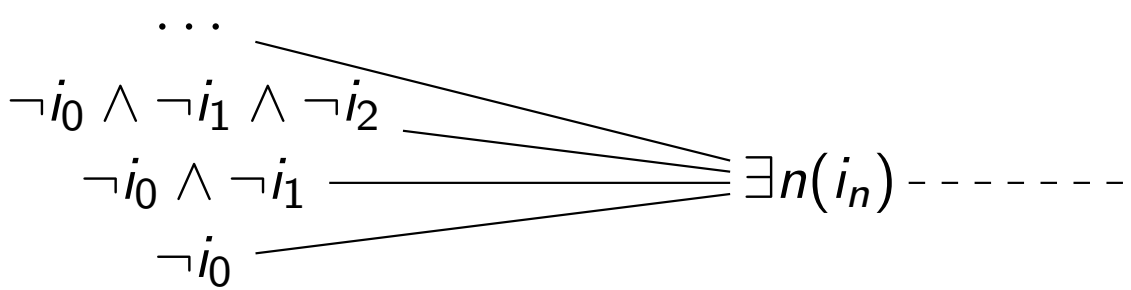
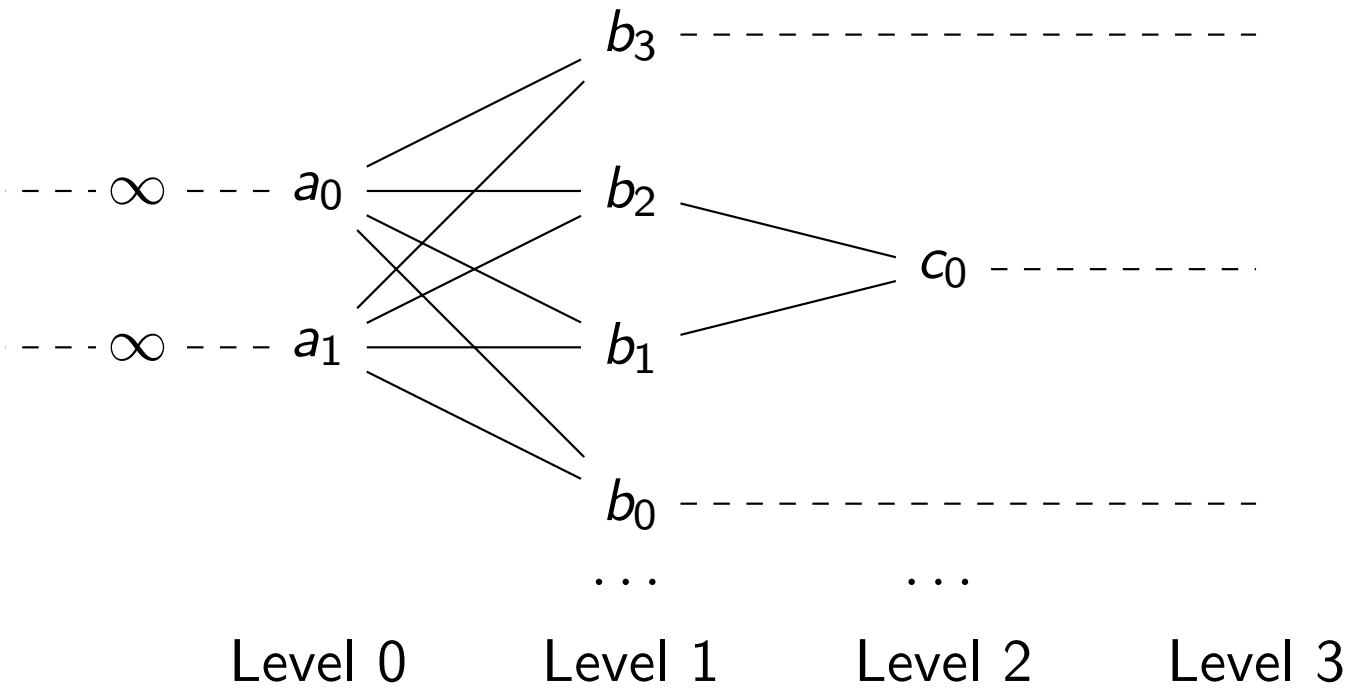
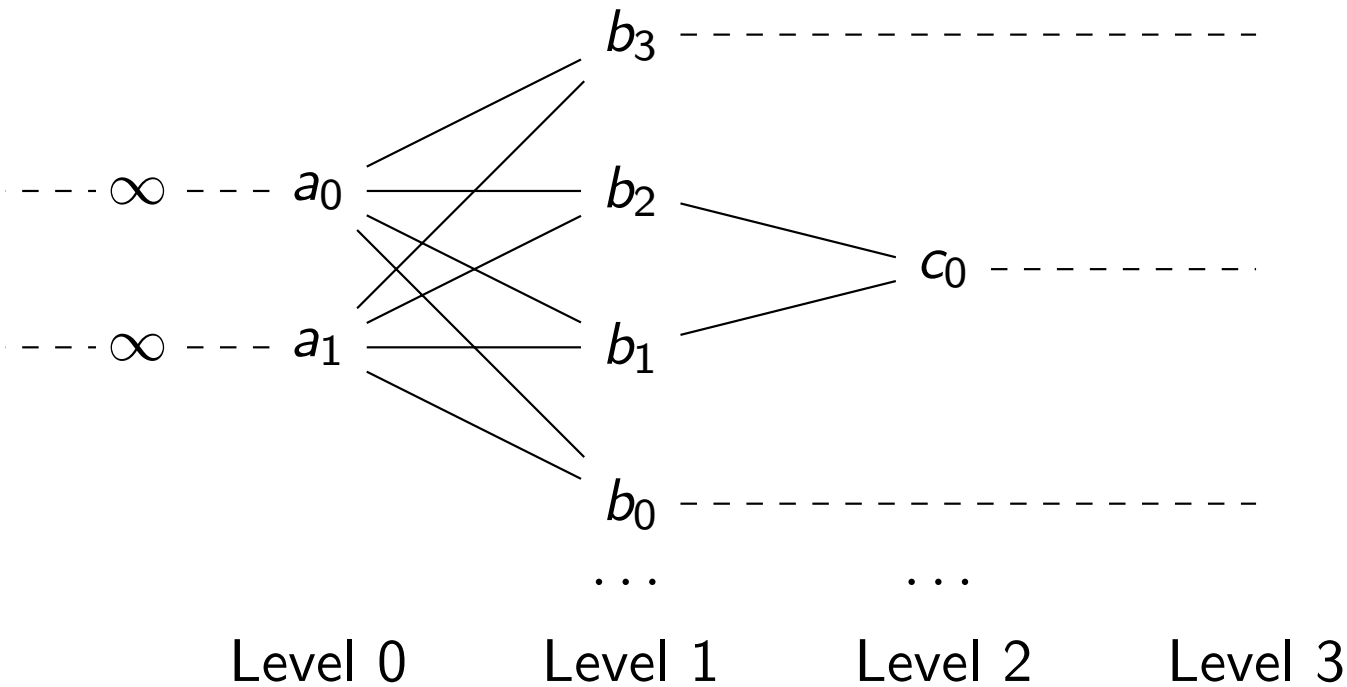
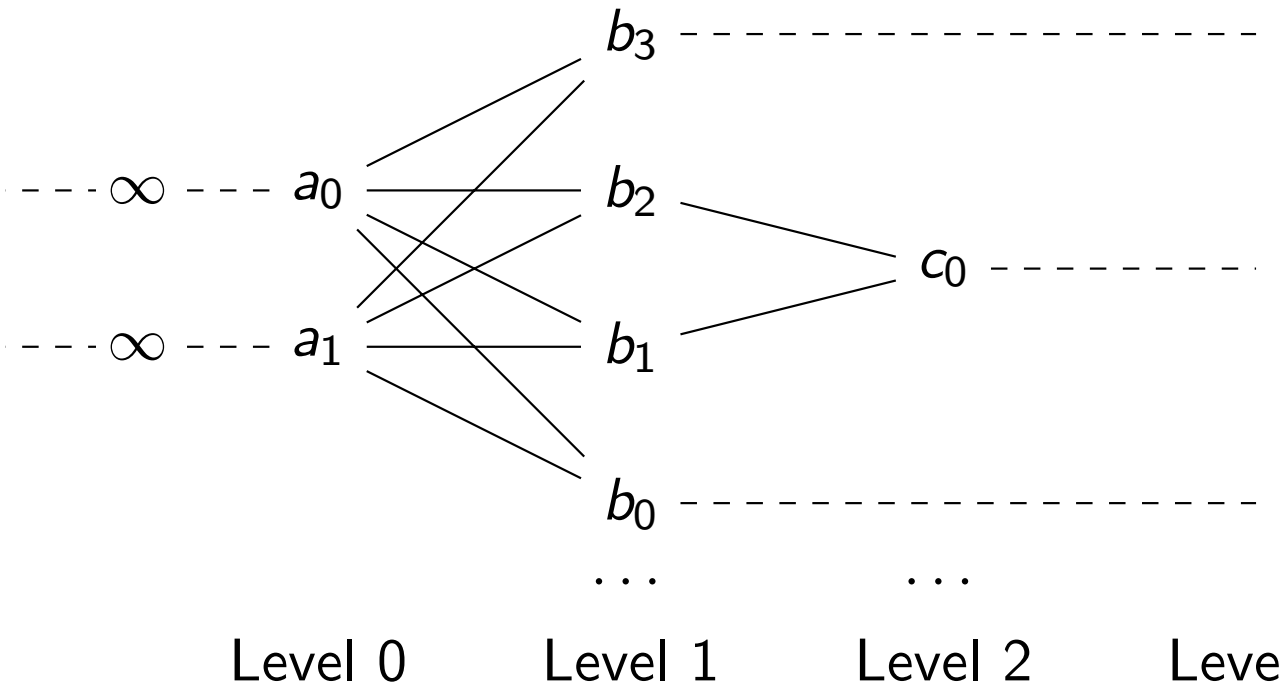


Now let's start with the construction of the graph.


b_3
∞
a_0
b_2
c_0
∞
a_1
b_1
b_0
...
...
Level 0
Level 1
Level 2
Level 3



Assume a_0 and a_1 are in the same partition, say 0 (we postpone the discussion on how to ensure this for later).



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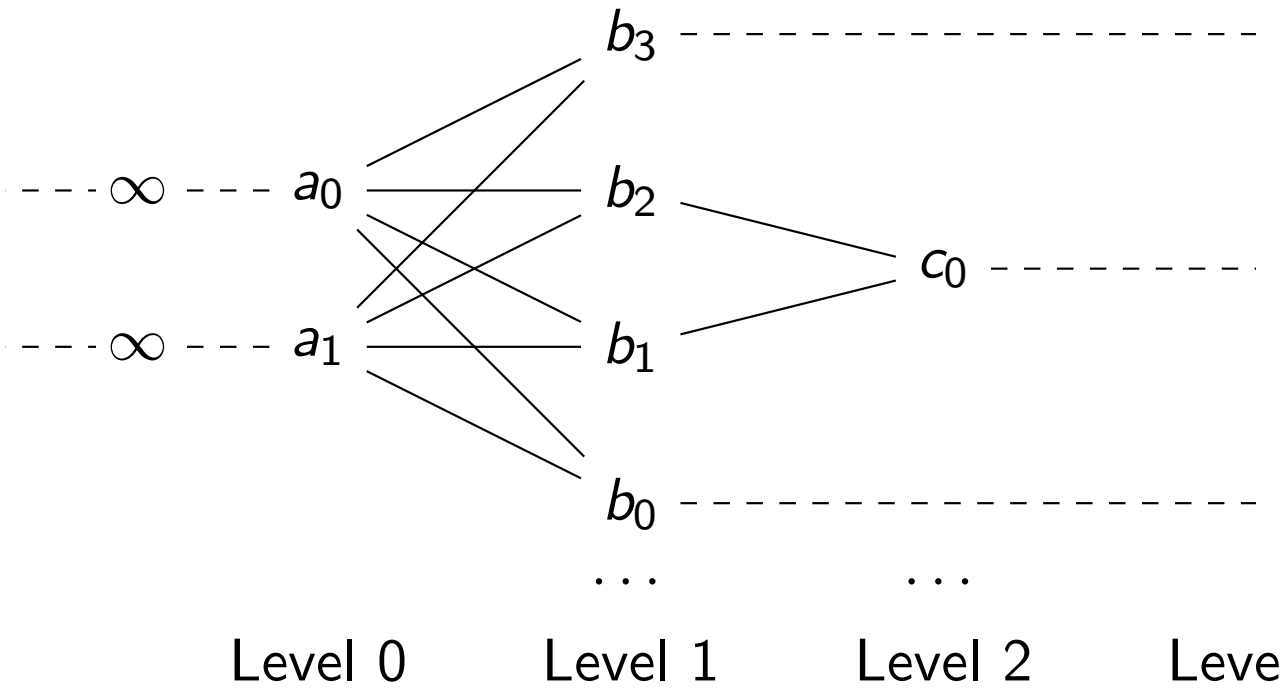
N.B.: the b_i's must be in partition 1 and the c_i's in partition 0.



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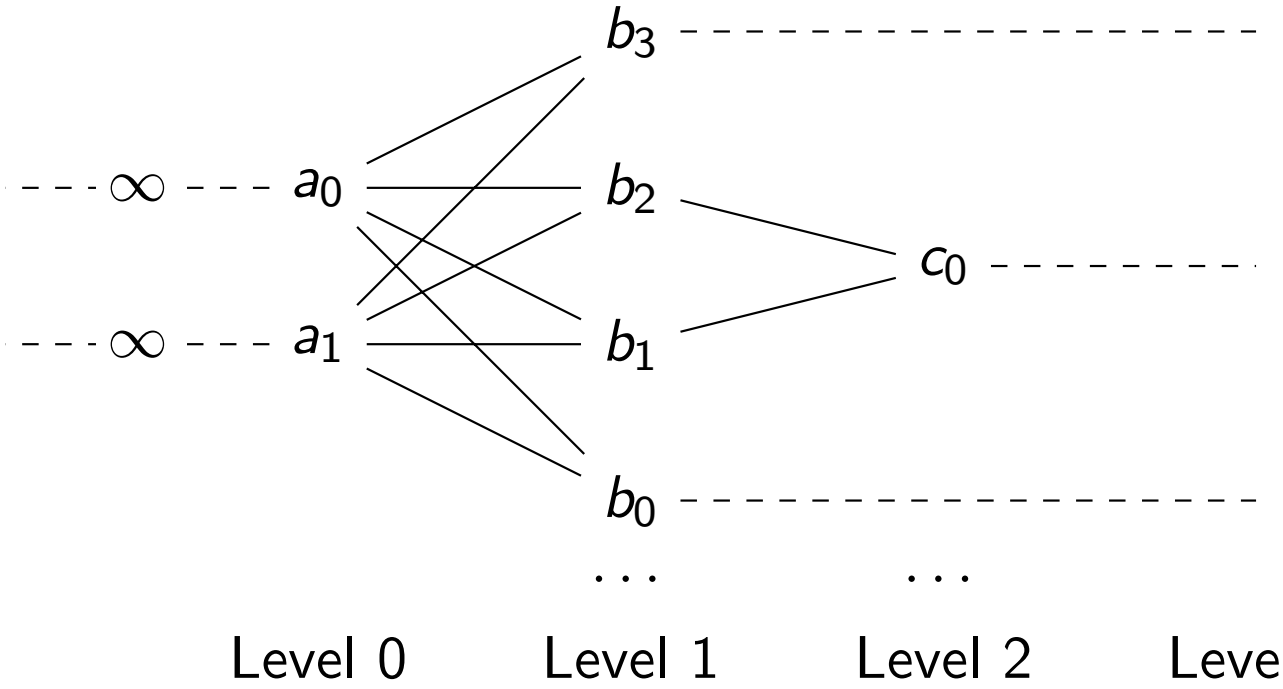
- c_i's are *cousins* of each other;
- b_{3n}'s are *cousins* of each other;
- c_i and b_{3n} are not *cousins*.

N.B. cousins are always on the same level and b_{3n+1} and b_{3n+2} are not cousins: they only force the c_i's to be in partition 0.



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Adding only one neighbor to c_i does not interfere with the partition in which c_i is.

$\Longrightarrow$ we will add to c_i a new neighbor, namely an output vertex of some logic gate.

The same reasoning can be applied to the b_{3n}'s.

To ensure that a_0 and a_1 are in the same partition, we consider instead a_0, a_1 and a_2: such vertices are at the level 0 of three graphs G_{01}, G_{02} and G_{12}.

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To compute the desired hyperarithemtic set we will only consider the u.p. of G_{ij} and forget about the other two.

We stop the description of the graph (for a slide) to recall what is a *Harrison order*[2].

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The order type of such an order is $\omega_1^{\mathsf{ck}}(1+\eta)+\alpha$ where:

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Starting from the setup described before, our graph will consist of levels G_α indexed by elements α of a fixed Harrison order.

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So far we have just described levels 0, 1, and 2. Notice that nodes at higher levels will be inputs and outputs of different gates.

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  - if v is at a *successor* level it will have 2 or 6 neighbors from lower levels (i.e., v is the output of some gate);
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No node has neighbors on the same level.

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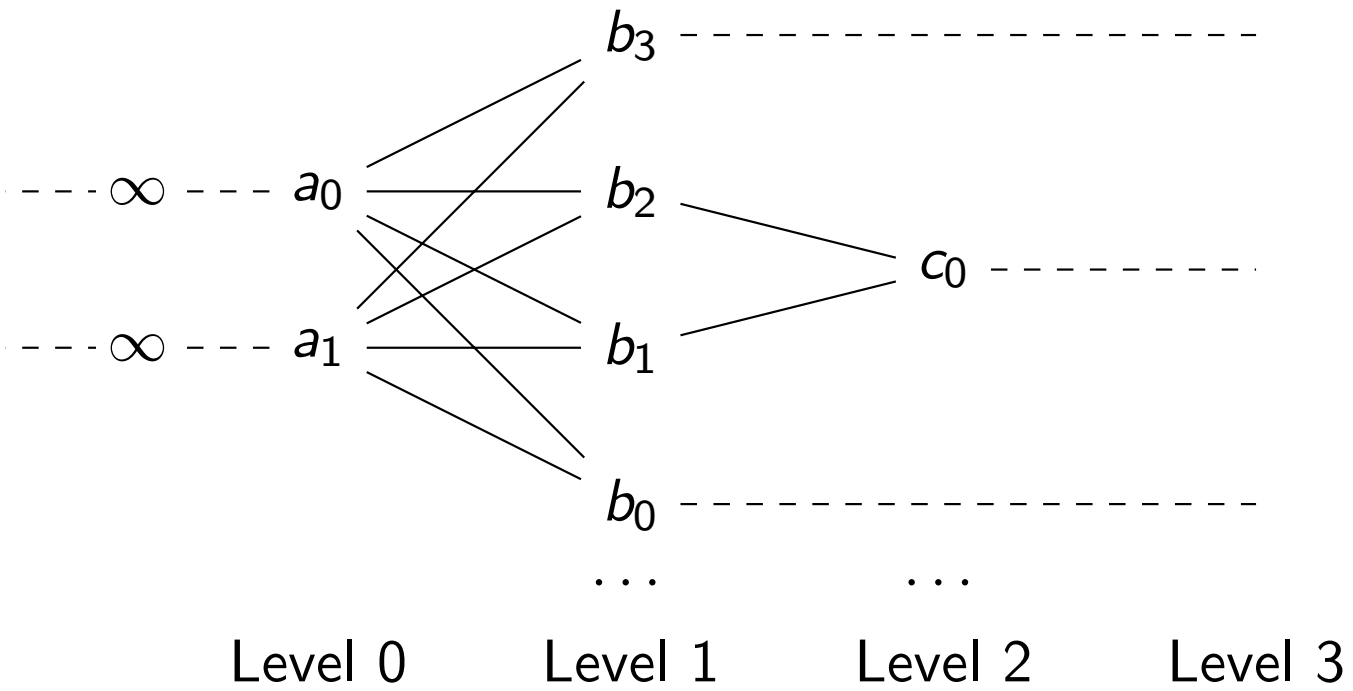

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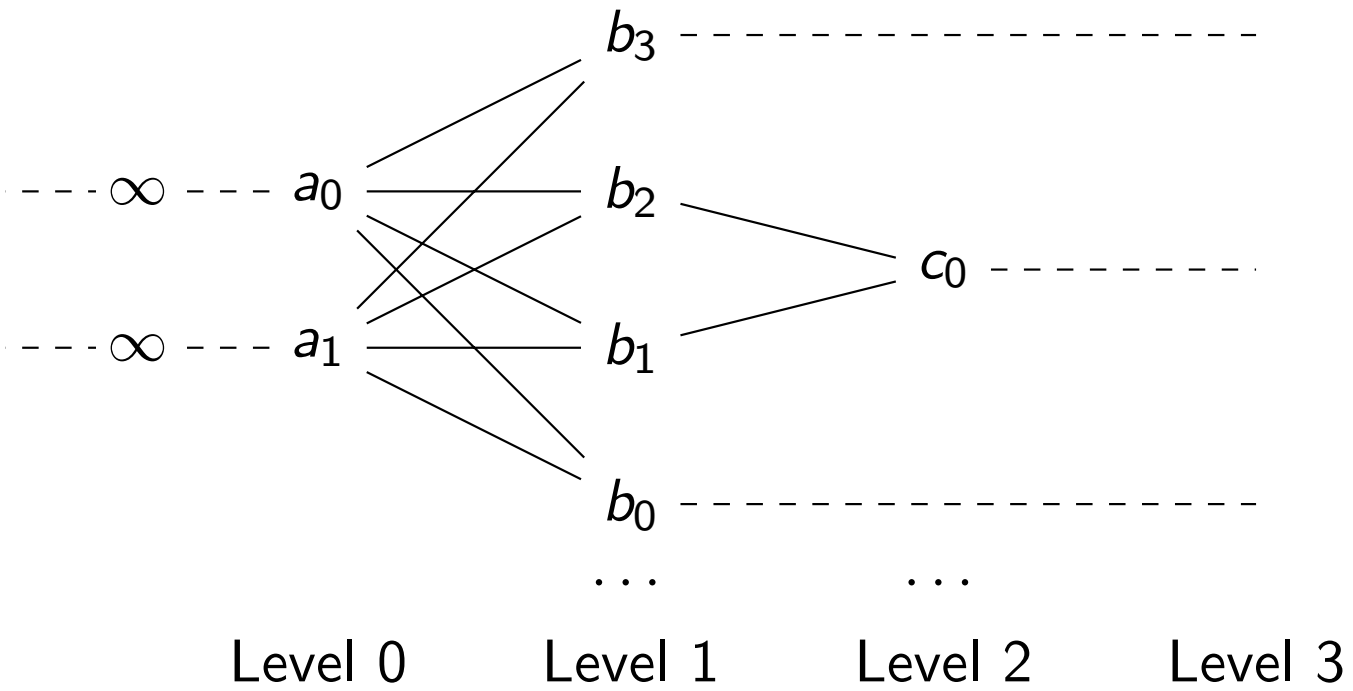

We know what gates are.

Key properties of cousins: $\alpha + 1$

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- Such formulae "approximate" the members in the desired hyperarithmetic set (recall how the existential gate worked: wait for the next slide).
- We need infinitely many because we may want to use that node/formula as an input for different gates. Indeed, remember that any node is both an input and an output for different gates. Using the same node as an input for different gates may "break" the gate of which it was an output.

Key properties of cousins: α limit

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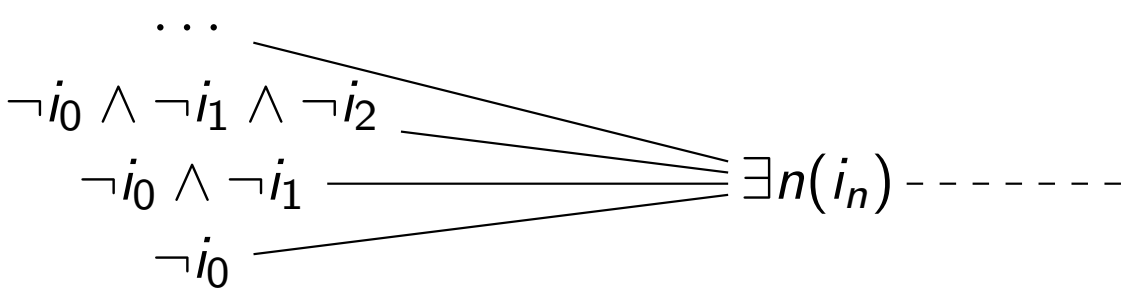
  How? Recall that vertices at limit levels have infinitely many neighbors coming from lower levels. These infinitely many nodes represent formulae approximating the fact that i is in the set or not: it will be in the set iff cofinitely many of its neighbors from lower levels are in partition 0.



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Done?

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Theorem

No graph like the one described above has a subdivision of a 5*-clique as a subgraph. In particular, by Berger's theorem, it has an unfriendly partition.*

An adaptation of the graph above allows us to prove the following (recall that by Bruhn and Diestel's theorem, any rayless graph admits an unfriendly partition).

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There is an "easiest" u.p. of this graph which is Turing reducible to all other u.p.: its Turing degree is between the α-th and $\alpha + 2$-nd jump of some computable set.

A universal graph

A "universal graph" for u.p.

Let $(\varphi_i)_{i\in\mathbb{N}}$ be a computable enumeration of all partial computable functions computing graphs.

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Let

$$V(G) = \{(e,k) : (\forall i,j \leq k)(\varphi_e(i,j)\downarrow \in \{0,1\})\}$$

and let

$$\big((e_i,k_i),(e_j,k_j)\big) \in E(G) \iff e_i = e_j \wedge k_i \neq k_j \wedge \varphi_{e_i}(k_i,k_j) = 1.$$

Notice that G is computable.

Theorem

Every computable graph is isomorphic to a component of G via a computable isomorphism. Furthermore, every u.p. of G is also a u.p. of every computable graph.

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To prove the theorem, given a graph H_e identified by φ_e, map any $v \in V(H_e)$ to the unique $f(v)$ s.t. $e_{f(v)} = e$ and $k_{f(v)} = v$.

Known results

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Theorem (Gandy basis theorem)

Every nonempty Σ^1_1 class has a member f such that $\omega_1^f = \omega_1^{\mathsf{ck}}$.

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The following definitions are due to Calvert, Franklin and Turetsky.

Theorem (High for isomorphism/paths)

We call a degree $\mathbf{d}$ *high for isomorphism if for any two computable structures $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \cong \mathcal{N}$, there is a* $\mathbf{d}$*-computable isomorphism from $\mathcal{M}$ to $\mathcal{N}$.*
We call a degree $\mathbf{d}$ *high for paths if for any ill-founded tree, the degree* $\mathbf{d}$ *computes a path in it.*

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The two notions coincide.

Theorem

Suppose that every computable graph has an u.p.: then there is no function f mapping a computable tree T to a computable graph G such that every u.p. of G computes some $p \in [T]$.

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Sketch.

Suppose such an f exists: then any u.p. A of the universal graph computes a path in every computable tree and hence A is high for isomorphism/paths.

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According to results of Calvert, Franklin and Turetsky $A''' \geq \mathcal{O}$ (where $\mathcal{O}$ is Kleene's $\mathcal{O}$).
So A should compute a well-ordered isomorphic to ω_1^{ck}. But, by Gandy basis theorem, one can choose A being low for ω_1^{ck} (i.e., A computes only computable ordinals). □

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Thanks!