Subrecursive degrees of representations of irrational numbers outside the cone of Cauchy sequences

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Our main question: *can we transform one representation into another without using unbounded search*?

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We will also denote

$$R_1 \equiv_S R_2 \ \text{ if } \ R_1 \preccurlyeq_S R_2 \ \& \ R_2 \preccurlyeq_S R_1$$

$$R_1 \prec_S R_2 \ \text{ if } \ R_1 \preccurlyeq_S R_2 \ \& \ R_2 \not\preccurlyeq_S R_1.$$

How to prove $R_1 \not\preceq_S R_2$?

The usual way is to construct an irrational $\alpha \in (0,1)$, such that:

- α has at least one R_2-representation in a subrecursive class $\mathcal{S}$;
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This works as long as $\mathcal{S}$ has sufficiently nice closure properties.

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Clearly $\mathcal{W} \preccurlyeq_S \mathcal{C}$: $W(n) = (C(n) - 2^{-n}, C(n) + 2^{n})$.

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But $\mathcal{C} \not\preccurlyeq_S \mathcal{W}$. Intuition: an unbounded search for n is needed to find $W(n)$, having length less than 2^{-n}. Formal proof: construct a computable $\alpha \in (0,1)$ by diagonalizing against all Cauchy sequences in a subrecursive class $\mathcal{S}$.

More representations

the Dedekind cut of α is $D : \mathbb{Q} \to \{0, 1\}$, where

$$D(q) = \begin{cases} 0, & \text{if } q < \alpha, \\ 1, & \text{if } q > \alpha. \end{cases}$$

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the continued fraction of α is $c : \mathbb{N} \to \mathbb{N}^+$, such that

$$\alpha = 0 + \cfrac{1}{c(0) + \cfrac{1}{c(1) + \cfrac{1}{\ddots}}}.$$

We will also denote $c = [\,]$.

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Conversely, we can combine D, $T^{\uparrow}$, $T^{\downarrow}$ to produce T.
It turns out that trace functions have the same degree as continued fractions: $T \equiv_S [\,]$.

Left and right best approximations

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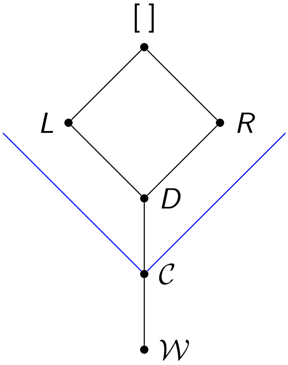
Picture of some known degrees



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Therefore, the degree of $\mathcal{W}$ is the only one, which might lead to new degrees, when combined with trace functions from below or from above.

The degrees of $\mathcal{W}^{\uparrow}$, $\mathcal{W}^{\downarrow}$, $\mathcal{W}^{\uparrow\downarrow}$

Let us define:

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Proof. Follows from the corresponding result in the presence of the Dedekind cut D.

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Proof. Diagonalize against all continued fractions in $\mathcal{S}$.

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Proof. Diagonalize against all continued fractions in $\mathcal{S}$.

It is known that any α with bounded continued fraction satisfies $|\alpha - p/q| > C/q^2$, therefore α has trace functions from below and from above in the class $\mathcal{S}$.

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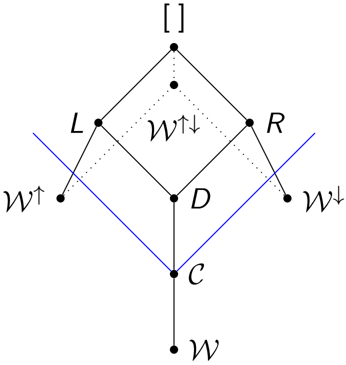
Corollary

$$\mathcal{C} \not\preceq_S \mathcal{W}^{\uparrow\downarrow}$$

Picture with the new degrees



Disjunction of the new degrees with $\mathcal{C}$

We know that the degrees $\mathcal{C} \vee \mathcal{W}^{\uparrow\downarrow}$, $\mathcal{C} \vee \mathcal{W}^{\uparrow}$ and $\mathcal{C} \vee \mathcal{W}^{\downarrow}$ lie strictly below $\mathcal{C}$.

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Proof. We can diagonalize against Cauchy sequences in $\mathcal{S}$ fast enough, so that no trace function from below or from above exists in $\mathcal{S}$ for the constructed number.

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Corollary

The four degrees $\mathcal{C} \vee \mathcal{W}^{\uparrow} \vee \mathcal{W}^{\downarrow}$, $\mathcal{C} \vee \mathcal{W}^{\uparrow}$, $\mathcal{C} \vee \mathcal{W}^{\downarrow}$ and $\mathcal{C} \vee \mathcal{W}^{\uparrow\downarrow}$ lie strictly between the degrees of $\mathcal{W}$ and $\mathcal{C}$.

Open question

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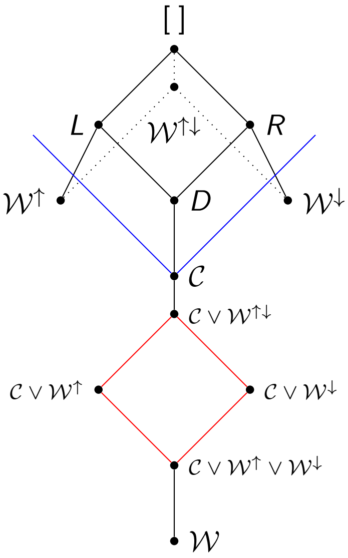
Conjecture

$$\mathcal{C} \vee \mathcal{W}^{\uparrow} \not\leq_S \mathcal{W}^{\downarrow}, \;\; \mathcal{C} \vee \mathcal{W}^{\downarrow} \not\leq_S \mathcal{W}^{\uparrow}$$

Final picture



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Thanks for your attention!