Towards Numerical Stability Analysis via Universal Envelopes
(work in progress)

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Black-Box Abstraction

- Idea: build complex programs from simpler ones.
- View programs as “black boxes” that satisfy some formal specification.
- Compose these “black boxes” to make new “black boxes”.
- In this talk, I will refer to “black boxes” as *modules*.

Black-Box Abstraction via Computable Functions

- Module: (realiser of a) computable (multi-valued) function.
- Composition rule: type of domain and co-domain match.

Problem (?):

- Over continuous data, this does not capture everything that practitioners consider possible.
- Numerical analysts provide "modules" for discontinuous (and hence uncomputable functions): QR-decomposition, singular value decomposition, finding Brouwer fixed points, etc.
- A solution to such a problem is an algorithm that produces a slightly perturbed solution to a slightly perturbed problem instance.
- Problem: this is not closed under composition.

Backwards Approximations

X, Y metric spaces.

$$f : X \to Y$$

The *backwards approximation of f* is

$$^{\dagger}f : X \times (0, +\infty) \rightrightarrows Y,$$
$$^{\dagger}f(x,\delta) = f(B(x,\delta)).$$

Fact. There exist $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f$ is continuous, but $g(f(x,\delta),\delta) \not\to g \circ f(x)$ as $\delta \to 0$.

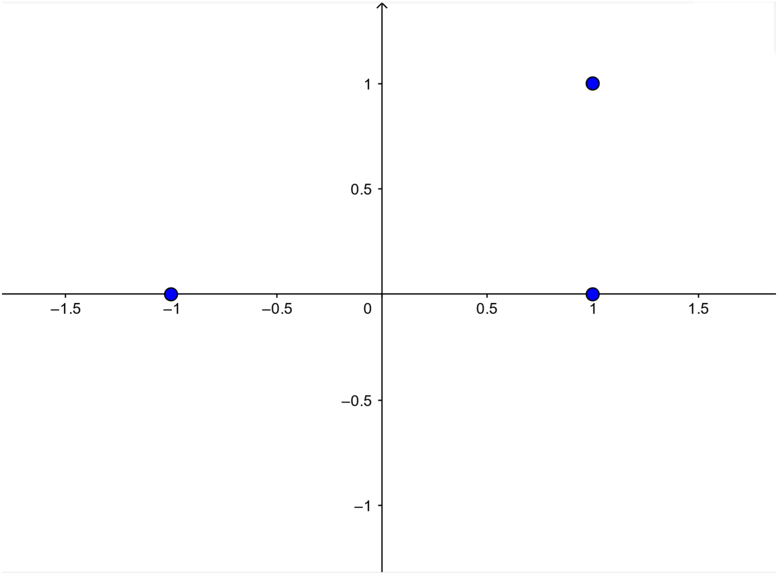
"Realistic" Example

- Goal: locate the convex hull of a set of points (with real number coordinates) in the plane.
- This is a computable problem.
- Computing the convex hull in the sense of Computational Geometry: extract from the points a list of points representing a polygon that bounds the convex hull.
- An "approximate" solution to the Computational Geometry problem allows us to locate the convex hull.
- How to find an approximate solution? Use an algorithm from Computational Geometry!

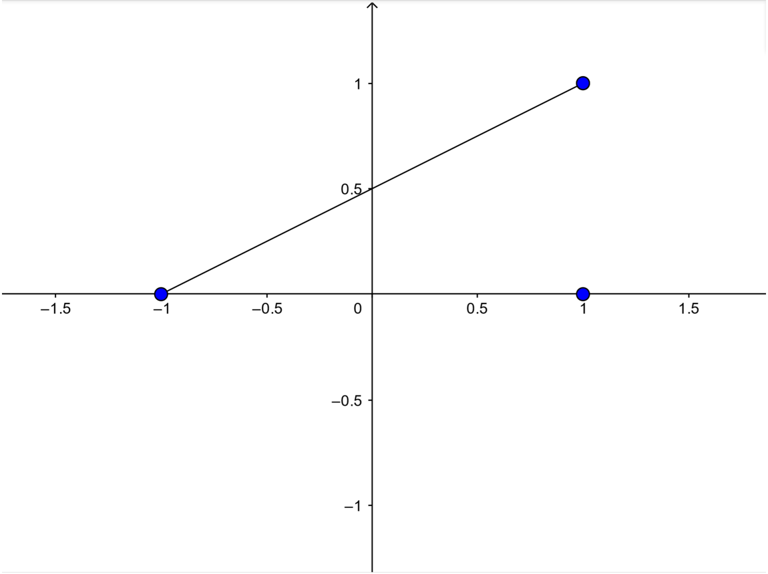
Jarvis Wrap

```
public static List<Point> ConvexHull (Point[] points)
{
    Point leftmost = points[0];
    for (int i = 0; i < points.Length; i++)
        if (points[i].x < leftmost.x)
            leftmost = points[i];

    List<Point> output = new List<Point>(points.Length){leftmost};

    bool done = false;
    Point lastPointAdded = leftmost;
    for (int i = 0; i < points.Length; i++)
    {
        if (!done)
        {
            Point endpoint = points[0];
            for (int j = 0; j < points.Length; j++)
            {
                if (endpoint == lastPointAdded
                || Orientation(lastPointAdded, endpoint, points[j]) == 1)
                    endpoint = points[j];
            }
            output.Add(endpoint);
            lastPointAdded = endpoint;
            done = endpoint == leftmost;
        }
    }
    return output;
}
```

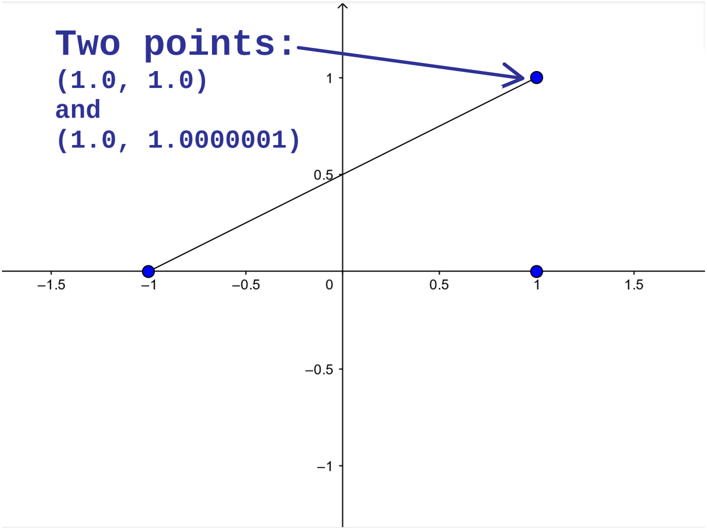
A Set of Points



The Output of the Algorithm



The Output of the Algorithm



High Level Questions

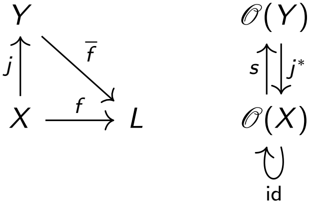
- What should a “module” look like in the context of “approximate computation”?
- What are the rules for composing modules?

Idea

- Basic modules: functions that compute backwards approximations + universal envelopes
- Composition rule: modules are composable if the composition of the universal envelopes is a universal envelope of the composition.
- Modules: elements of the closure of the basic modules under the composition rule.

Injective Spaces

A QCB-space L is (Σ-split) injective if every continuous map $f\colon X \to L$ extends continuously along every Σ-split embedding $j\colon X \to Y$:



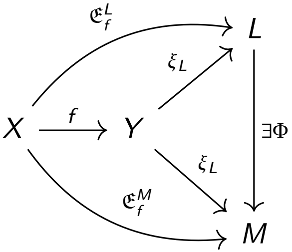
Fact. Injective spaces are simultaneously $\mathscr{K}$-algebras and $\mathscr{V}$-algebras, and hence complete lattices.

Envelopes

Let $f\colon X \to Y$ be any map.
Let L be an injective space with a Σ-split embedding $\xi_L\colon Y \to L$.
The L-envelope of f is the greatest continuous map $\mathfrak{E}^L_f\colon X \to L$ satisfying

$$\mathfrak{E}^L_f \leq \xi_L \circ f.$$

Tightening:



$$\Phi \circ \mathfrak{E}^L_f = \mathfrak{E}^M_g$$

$$\Phi \circ \xi_L = \xi_M$$

An envelope of f is called universal if it tightens all envelopes of f.

Envelopes and Backwards Approximations

Theorem (N. 2022)

Let $X_1, \dots, X_{n+1}$ be a finite sequence of computable metric spaces. Let $f_i \colon X_i \to X_{i+1}$, $i = 1, \dots, n$. For $i = 1, \dots, n$, let $F_i \colon X_i \to \mathscr{K}_\perp(X_{i+1})$ be the $\mathscr{K}_\perp(X_{i+1})$-envelope of f_i. Assume that $F_i(x) \neq \perp$ for all $x \in X_i$. Then the following are equivalent:

- *For all $x \in X_1$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $y \in {}^\dagger f_n(\cdot, \delta) \circ \dots \circ {}^\dagger f_1(\cdot, \delta)(x)$ we have $d\left(y, f_n \circ \dots \circ f_1(x)\right) < \varepsilon$. Moreover, this convergence is uniform on compact sets*
- *We have $F_n \circ \dots \circ F_1(x) = \{f_n \circ \dots \circ f_1(x)\}$ for all $x \in X_1$. Here, the composition of the F_i's is taken in the Kleisli category of the monad $\mathscr{K}_\perp$.*

Backwards Approximations: a Flaw in the Definition

- The backwards approximations above do not capture everything we can and should do...
- Consider

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} -x & \text{if } x < 0, \\ 1 & \text{otherwise.} \end{cases}$$

- $f \circ f = 1$.
- The $\mathscr{K}_\perp(\mathbb{R})$ envelope is

$$F(x) = \begin{cases} \{-x\} & \text{if } x < 0, \\ \{0,1\} & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

- $F \circ F(0) = \{0,1\} \neq \{f \circ f(0)\}$.
- By the theorem, ${}^\dagger f({}^\dagger f(0,\delta),\delta)$ does not converge to 1 as $\delta \to 0$.
- Problem: $f(B(0,\delta))$ contains arbitrarily small numbers.

More general set up

Consider functions

$$f : X \to U \subseteq Y$$

where:

- X and Y are computable metric spaces.
- U is an open subset of Y satisfying $f(X) \subseteq U$.
- Idea: U represents "observable constraints" on the values of f.

Envelopes and Backwards Approximations

Let

$$f : X \to U \subseteq Y$$

as above.

- An *envelope* of f is an envelope of the function $f : X \to U$.
- "The" *backwards approximation* of f is the multi-valued map

$$^{\dagger}f : X \times (0,\eta) \rightrightarrows Y,\ ^{\dagger}f(x,\delta) = f(B(x,\delta)) \setminus B(Y \setminus U,\delta)$$

  where $\eta > 0$ is chosen such that $^{\dagger}f(x,\delta) \neq \emptyset$ for all x and $\delta \in (0,\eta)$.
- A backwards approximation need not exist – we will consider only functions where this does exist.

First Example

$$f : \mathbb{R} \to (0, +\infty) \subseteq \mathbb{R},\ f(x) = \begin{cases} -x & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

- Universal envelope:

$$\mathfrak{E}_f : \mathbb{R} \times \mathbb{R} \to \mathscr{K}_\perp(\mathbb{R}),\ \mathfrak{E}_f(x) = \begin{cases} \{-x\} & \text{if } x < 0 \\ \{0, 1\} & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

- This is **not** a universal envelope of $f : \mathbb{R} \to \mathbb{R}$.
- The backwards approximation with error δ must produce a number $\geq \delta$.
- Hence ${}^\dagger f({}^\dagger f(x, \delta), \delta) = 1$ for all x.

Example: Labelling Real Numbers with Equality Information

$$\ell_{=?} : \mathbb{R} \times \mathbb{R} \to ((\mathbb{R} \times \mathbb{R}) \setminus \Delta_{\mathbb{R}}) + \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R} + \mathbb{R}.$$

- Universal envelope:

$$\mathfrak{E}_{\ell_=} : \mathbb{R} \times \mathbb{R} \to \mathscr{K}_\perp(\mathbb{R} \times \mathbb{R} + \mathbb{R}),\ \mathfrak{E}_{\ell_=}(x,y) = \begin{cases} \{(0,\{x,x\}),(1,\{x\})\} & \text{if } x = y \\ \{(0,\{x,y\})\} & \text{if } x \neq y \end{cases}$$

- This is **not** a universal envelope of $\ell_{=?} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} + \mathbb{R}$.
- Generalisation: turning a list of objects into a multiset (*e.g.* count multiplicities of roots, eigenvalues etc.).

Example: Labelling Real Numbers with Equality Information

$$\ell_{=?} : \mathbb{R} \times \mathbb{R} \to \big((\mathbb{R} \times \mathbb{R}) \setminus \Delta_{\mathbb{R}}\big) + \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R} + \mathbb{R}.$$

- Backwards approximation takes $x, y \in \mathbb{R}$, $\delta > 0$ and returns:
  - Either a number $z \in \mathbb{R}$ with $|x - z| < \delta$ and $|y - z| < \delta$
  - or numbers $\tilde{x}, \tilde{y} \in \mathbb{R}$ with $|x - \tilde{x}| < \delta$, $|y - \tilde{y}| < \delta$, and $|\tilde{x} - \tilde{y}| \geq 2\delta$

Example: Division

$$\mathrm{div}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}\cup\{\pm\infty,\mathrm{NaN}\}\subseteq\mathbb{R}\cup\{\pm\infty,\mathrm{NaN}\},$$

$$\mathrm{div}(x,y)=\begin{cases} x/y & \text{if } y\neq 0,\\ \pm\infty & \text{if } y=0, x\neq 0\\ \mathrm{NaN} & \text{if } y=0, x=0\end{cases}$$

- Universal envelope:

$$\mathfrak{E}_{\mathrm{div}}\colon \mathbb{R}\times\mathbb{R}\to\mathscr{K}_{\perp}(S^1\cup\{\pm\infty,\mathrm{NaN}\}),$$

$$\mathfrak{E}_{\mathrm{div}}=\begin{cases} \Phi(\{x/y\}) & \text{if } y\neq 0,\\ \{N\}\cup\{\pm\infty\} & \text{if } y=0, x\neq 0\\ S^1\cup\{\mathrm{NaN},\pm\infty\} & \text{if } y=0, x=0\end{cases}$$

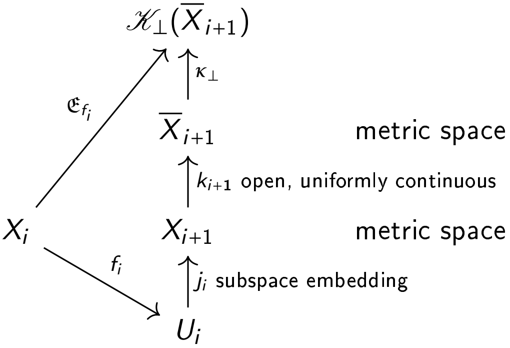
where $\Phi\colon \mathbb{R}\to S^1$ is the inverse stereographic projection, and $N\in S^1$ is the north pole.

Set Up

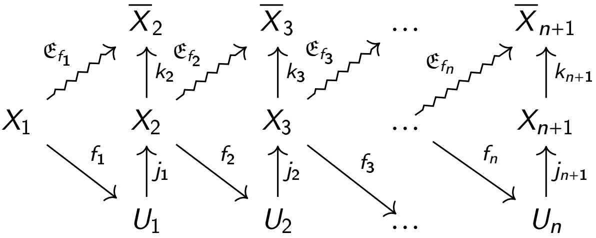
Consider maps

$$f_i \colon X_i \to U_i \subseteq X_{i+1}$$

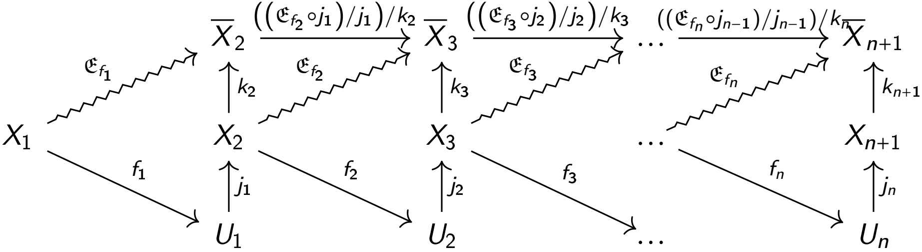
with universal envelopes of the form



Set Up


$\overline{X}_2$
$\overline{X}_3$
$\dots$
$\overline{X}_{n+1}$
$\mathfrak{E}_{f_1}$
$\mathfrak{E}_{f_2}$
$\mathfrak{E}_{f_3}$
$\mathfrak{E}_{f_n}$
k_2
k_3
k_{n+1}
X_1
X_2
X_3
$\dots$
X_{n+1}
f_1
f_2
f_3
f_n
j_1
j_2
j_{n+1}
U_1
U_2
$\dots$
U_n

Set Up



$((\mathfrak{E}_{f_2} \circ j_1)/j_1)/k_2$
$((\mathfrak{E}_{f_3} \circ j_2)/j_2)/k_3$
$((\mathfrak{E}_{f_n} \circ j_{n-1})/j_{n-1})/k_n$
$\overline{X}_2$
$\overline{X}_3$
$\overline{X}_{n+1}$
$\mathfrak{E}_{f_1}$
$\mathfrak{E}_{f_2}$
$\mathfrak{E}_{f_3}$
$\mathfrak{E}_{f_n}$
k_2
k_3
k_{n+1}
X_1
X_2
X_3
X_{n+1}
f_1
f_2
f_3
f_n
j_1
j_2
j_n
U_1
U_2
U_n

Theorem

Let $f_1,\ldots,f_n$ *be functions as above, and let* $F_i\colon X_i \to \mathscr{K}_\perp\left(\overline{X}_{i+1}\right)$ *be a universal envelope of* f_i *for all* i. *If*

$$(((F_n \circ j_{n-1})/j_{n-1})/k_{n-1})_* \circ \cdots \circ (((F_2 \circ j_1)/j_1)/k_1)_* \circ F_1(x)$$
$$= \{k_n \circ j_n \circ f_n \circ j_{n-1} \circ \cdots \circ f_2 \circ j_1 \circ f_1(x)\}$$

then for all $x \in X_1$*:*

$$^\dagger f_n(\cdot,\delta) \circ \cdots \circ {}^\dagger f_2(\cdot,\delta) \circ {}^\dagger f_1(x,\delta) \to f_n \circ \cdots \circ f_2 \circ f_1(x)$$

as $\delta \to 0$. *Moreover, this convergence is uniform on compact sets.*

The converse direction does not hold!

Counterexample

- Let $f_1(x) = \frac{x}{(x-1)^2(x+1)^2}$.
- Extend this to a function $f_1 : \mathbb{R} \to \mathbb{R}$ by letting $f(1) = 1$ and $f(-1) = -1$.
- Let $f_2(x) = x$.
- Let

$$f_3(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

- Then

$$^{\dagger}f_3(\cdot,\delta) \circ {}^{\dagger}f_2(\cdot,\delta) \circ {}^{\dagger}f_1(x,\delta) \to f_3 \circ f_2 \circ f_1(x)$$

  as $\delta \to 0$, uniformly on compact sets.

Counterexample

Universal envelopes:

$$F_1 : \mathbb{R} \to \mathscr{K}_\perp([-\infty, +\infty]),\ F_1(x) = \begin{cases} \{k \circ f_1(x)\} & \text{if } x \notin \{1, -1\}, \\ \{1, +\infty\} & \text{if } x = 1, \\ \{-1, -\infty\} & \text{if } x = -1. \end{cases}$$

$$F_2 : \mathbb{R} \to \mathscr{K}_\perp(\mathbb{R}),\ F_2(x) = \{x\}.$$

$$F_3 : \mathbb{R} \to \mathscr{K}(\mathbb{R}),\ F_3(x) = \begin{cases} \{-1, 1\} & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0, \\ \{1\} & \text{if } x > 0. \end{cases}$$

Counterexample

We have

$$(F_2/k_2)(+\infty) = \bot$$

so

$$F_3 \circ (F_2/k_2) \circ F_1(1) = \bot.$$

So this "composition" of the universal envelopes is not a universal envelope of the composition.

Co-Envelopes

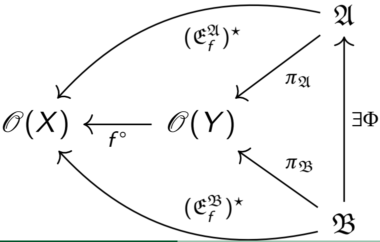
An *advice bundle* over Y is an injective space $\mathfrak{A}$ together with a frame homomorphism $\pi_{\mathfrak{A}}\colon \mathfrak{A} \to \mathscr{O}(Y)$ that admits a section $s\colon \mathscr{O}(Y) \to \mathfrak{A}$ (which we may take to be a lower adjoint).

Let $f\colon X \to Y$. Let $\mathfrak{A}$ be an advice bundle over Y. The $\mathfrak{A}$-co-envelope of f is the greatest continuous map $(\mathfrak{E}_f^{\mathfrak{A}})^\star\colon \mathfrak{A} \to \mathscr{O}(X)$ satisfying

$$(\mathfrak{E}_f^{\mathfrak{A}})^\star \leq f^\circ \circ \pi_{\mathfrak{A}}$$

where $f^\circ(U) = f^{-1}(U)^\circ$.

Tightening:

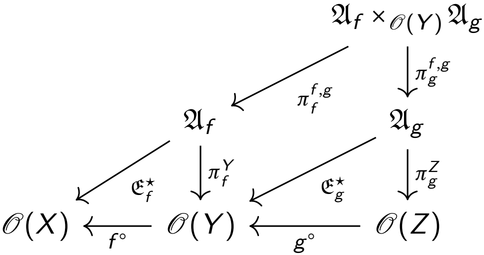


$$(\mathfrak{E}_f^{\mathfrak{A}})^\star \circ \Phi \geq (\mathfrak{E}_f^{\mathfrak{B}})^\star$$

$$\pi_{\mathfrak{A}} \circ \Phi = \pi_{\mathfrak{B}}$$

Composition of Co-Envelopes

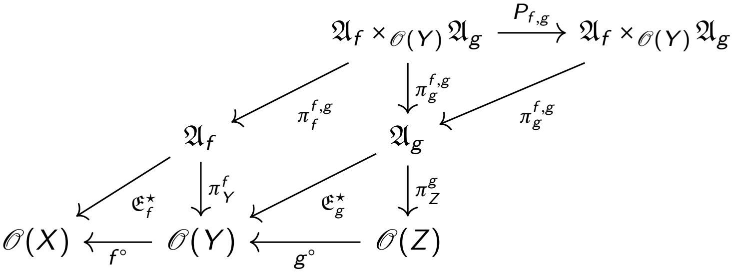
Pullback



$$\mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g = \left\{ (x,y) \in \mathfrak{A}_f \times \mathfrak{A}_g \mid \pi_Y^f(x) = \mathfrak{E}_g^\star(y) \right\}.$$

Intuition: an element of $\mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g$ consists of an open set $V \in \mathscr{O}(Z)$ together with some extra information α about V and some extra information about "$\mathfrak{E}_g^\star([V,\alpha])$".

Pullback



$$P_{f,g} : \mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g \to \mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g$$

greatest continuous map satisfying

$$\pi_g^{f,g} \circ P_{f,g} = \pi_g^{f,g}.$$

Intuition: $P_{f,g}$ takes an open set $V \in \mathscr{O}(Z)$ with some extra information α about V and some extra information β about "$\mathfrak{E}_g^\star([V,\alpha])$" and computes as much extra information β' as possible about "$\mathfrak{E}_g^\star([V,\alpha])$".

Composition

$$P_{f,g} : \mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g \to \mathfrak{A}_f \times_{\mathscr{O}(Y)} \mathfrak{A}_g$$

greatest continuous map satisfying

$$\pi_g^{f,g} \circ P_{f,g} = \pi_g^{f,g}.$$

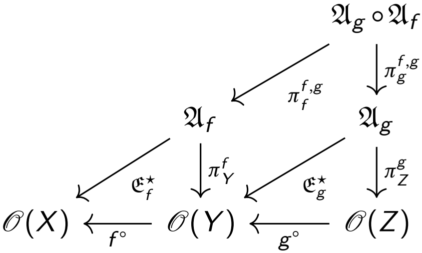
Composition

$$\mathfrak{A}_g \circ \mathfrak{A}_f = \{\text{fixed points of } P_{f,g}\} / \sim$$

where $(x,y) \sim (x',y')$ iff $\pi_Z^g(y) = \pi_Z^g(y')$ and $x = x'$.
Intuition:

- Take "maximal amounts of information" about open sets in Y computable from open subsets of Z with extra information.
- Identify those "bits of extra information" that yield the same open subset of Y and the same information about it.

Composition of Co-Envelopes



$$\mathfrak{E}_f^\star \bullet \mathfrak{E}_g^\star : \mathfrak{A}_g \circ \mathfrak{A}_f \to \mathscr{O}(X),\ \mathfrak{E}_f^\star \circ \mathfrak{E}_g^\star = \mathfrak{E}_f^\star \circ \pi_f^{f,g}$$

Facts about the Composition

- Letting $\leq$ denote the tightening relation, if $\mathfrak{E}^{\star}_{f,0} \leq \mathfrak{E}^{\star}_{f,1}$ and $\mathfrak{E}^{\star}_{g,0} \leq \mathfrak{E}^{\star}_{g,1}$ then $\mathfrak{E}^{\star}_{f,0} \bullet \mathfrak{E}^{\star}_{g,0} \leq \mathfrak{E}^{\star}_{f,1} \bullet \mathfrak{E}^{\star}_{g,1}$.
- Composition is associative.
- For uniform envelopes, this composition is the same thing as composition in the Kleisli category of $\mathscr{K}_{\perp}$.

Theorem

The following are equivalent:

- *There exist continuous multi-valued functions* $\omega_i :\subseteq X_i \times \mathbb{Q}_{>0} \rightrightarrows \mathbb{Q}_{>0}$, *with* ω_1 *total and*

$$(x_i, \varepsilon) \in \mathrm{dom}(\omega_i) \wedge x_{i+1} \in^{\dagger} f_i(x_i, \varepsilon) \to (x_{i+1}, \varepsilon) \in \mathrm{dom}(\omega_{i+1}),$$

  such that for all sequences $x_1, \dots, x_{n+1}$, $\delta_1 > 0, \dots, \delta_n > 0$ *satisfying* $\delta_i \in \omega_i(x_i, \varepsilon)$ *and* $x_{i+1} \in^{\dagger} f_i(x_i, \delta_i)$ *we have* $d(x_{n+1}, f_n \circ \dots \circ f_1(x)) < \varepsilon$.
- *The composition*

$$\mathfrak{E}^{\star}_{f_1} \bullet \dots \bullet \mathfrak{E}^{\star}_{f_n}$$

  is the universal co-envelope of $f_n \circ \dots \circ f_1$.

Main Open Problems

- We have two theorems connecting convergence of "approximate solutions" to composition of envelopes.
- There is a big "gap" between the two notions of convergence.
- How much can we close this gap? For example, can we find a notion of "approximate computation" that gives us stronger convergence in the setting of the second theorem?
- In the other direction, under which assumptions can we define the composition of envelopes, and reduce the situation to the first theorem?