The equational theory of the Weihrauch lattice with product and composition

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joint work with Eike Neumann and Arno Pauly

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Errata (27/08/24)

- $a \times (b \sqcap c) \leq a \times (b \sqcap (a \times c))$ is actually derivable
- For $(\mathfrak{W}, \sqcap, \times, 1)$, the proposed theory is complete iff the pointed one is
- Cycles in $\Game$ may actually have exponential size, so the "Corollary" did not follow from the conjecture as expected on slide 13.

Relevant items are striked out.

Most of the conjectures of part 2 now have proofs on arXiv.

`https://arxiv.org/abs/2403.13975`
`https://arxiv.org/abs/2408.14999`

Weihrauch problems

Definition

A Weihrauch problem P is given

- a set of instances $\mathsf{dom}(P) \subseteq \mathbb{N}^{\mathbb{N}}$
- for each $i \in \mathsf{dom}(P)$ a non-empty set of solutions $P_i \subseteq \mathbb{N}^{\mathbb{N}}$

Examples:

- $\mathsf{C}_{\mathbb{N}}$: "Given $p \in \mathbb{N}^{\mathbb{N}}$, find something not enumerated by p"

$\mathsf{dom}(\mathsf{C}_{\mathbb{N}}) = \{p \in \mathbb{N}^{\mathbb{N}} \mid \exists n\, n \notin \mathsf{range}(p)\}$ $\qquad$ $\mathsf{C}_{\mathbb{N}}(p) = \{1^n 0^\omega \mid n \notin \mathsf{range}(p)\}$

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Comparing the hardness of problems $\rightsquigarrow$ via a notion of reducibility

Weihrauch reducibility

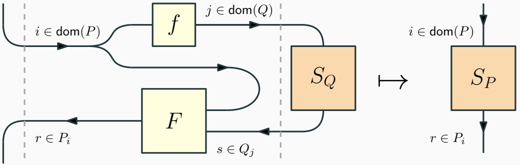
TL;DR: Turing reducibility, but

- adapted to type 2 computability
- reductions must make **exactly** one oracle call

Official definition

$P \leq_W Q$ if there are **computable**

$$f : \mathsf{dom}(P) \to \mathsf{dom}(Q) \quad \text{and} \quad F : \prod_{i \in \mathsf{dom}(P)} (Q_{f(i)} \to P_i)$$



Reductions compose + Quotienting by $\equiv_W$ $\rightsquigarrow$ **Weihrauch degrees**

Some operation on the Weihrauch degrees

Many natural operators over Weihrauch problems/degrees:

- Joins $\sqcup$: "solve either one of the problems"

$$\mathsf{dom}(P \sqcup Q) \cong \mathsf{dom}(P) + \mathsf{dom}(Q) \qquad \begin{array}{rcl} (P \sqcup Q)_{\mathsf{in}_1(i)} & = & P_i \\ (P \sqcup Q)_{\mathsf{in}_2(j)} & = & Q_j \end{array}$$

- Meets $\sqcap$: "given inputs for both, solve one"

$$\mathsf{dom}(P \sqcap Q) \cong \mathsf{dom}(P) \times \mathsf{dom}(Q) \quad (P \sqcap Q)_{i,j} = P_i + Q_j$$

- Products $\times$: "solve both problems"

$$\mathsf{dom}(P \times Q) \cong \mathsf{dom}(P) \times \mathsf{dom}(Q) \quad (P \times Q)_{i,j} = P_i \times Q_j$$

- 1: "there is an instance, everything is a solution"
- ...

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Driving question

Can we extend this to additional operations? In particular:

- Can we axiomatize equation in those extensions?
- What is the complexity of deciding universal validity of $t \leq u$?

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Known axioms investigated in "On the algebraic structure of Weihrauch degrees", Brattka & Pauly, LMCS 2018

$\sqcap, 1, \times, \sqcup$ and $(-)^*$ (preprint on arXiv)

A partial axiomatization of $(\mathfrak{W}, \sqcap, \times, 1)$

$a \le b \Rightarrow a \times c \le b \times c$	monotonicity
$a \times (b \times c) = (a \times b) \times c \qquad a \times 1 = a$	monoid structure
$a \times b = b \times a$	commutativity
$a \le a \times a$	relevance
$a \sqcap b \le a \qquad a \sqcap b \le b$	$\sqcap$ is a lower bound
$a \le b \ \wedge\ a \le c \ \Rightarrow\ a \le b \sqcap c$	$\sqcap$ is the greatest lb
$(a \times b) \sqcap c \le a \times (b \sqcap c)$	half-distributivity

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Open question

Adding $1 \leq a \Rightarrow$ complete axiomatization in the **pointed** degrees?

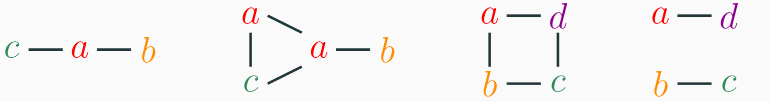
Terms as graphs

Goal to make things more tractable

Reduce checking $(\mathfrak{W}, \sqcap, \times) \models t \leq u$ to a combinatorial problem.

First step: interpret terms t as finite coloured graphs G_t:

- For a variable x, take a single vertex coloured by x.
- For $\sqcap$, take the disjoint union
- For $\times$, disjoint union + all edges between the two components

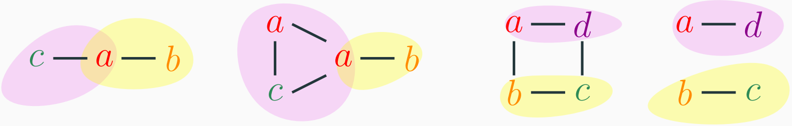


$a \times (c \sqcap b)$ $\quad$ $a \times ((a \times c) \sqcap b)$ $\quad$ $(a \sqcap c) \times (b \sqcap d)$ $\quad$ $(a \times d) \sqcap (b \times c)$

Combinatorial reduction between graphs

Definition

A reduction from (V_0, E_0, c_0) to (V_1, E_1, c_1) is a colour-preserving function $h : V_1 \to V_0$ such that the image of any maximal clique in (V_1, E_1) under h contains a maximal clique in (V_0, E_0).



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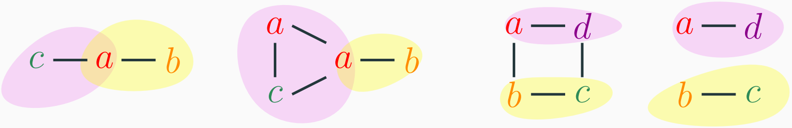

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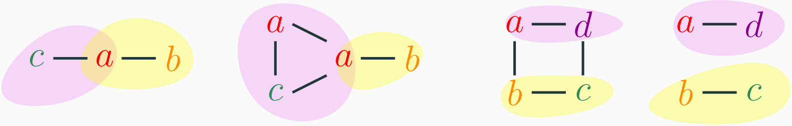

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Combinatorial characterization and complexity

$(\mathfrak{W}, \sqcap, \times) \models t \leq u$ iff there is a reduction from G_t to G_u.
As a result, deciding $(\mathfrak{W}, \sqcap, \times) \models t \leq u$ is Σ_2^p-complete.

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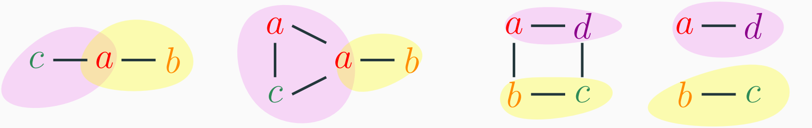

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As a result, deciding $(\mathfrak{W}, \sqcap, \times) \models t \leq u$ is Σ_2^p-complete.

Corollary: same valid inequations as the free $\sqcup$-$\sqcap$-completion of the ordered monoid $(\mathbb{N}_{>0}, |, 1, \times)$

Handling $\sqcup$ is easier

Relative completeness

True inequations in $(\mathfrak{W}, \sqcap, \times)$ + the axioms below derive all true inequations in $(\mathfrak{W}, \sqcup, \sqcap, \times, 1, (-)^*)$

(P^* is finite parallelization, the lfp of $X \mapsto 1 \sqcup X \times P$)

$$a \leq a \sqcup b \qquad b \leq a \sqcup b$$
$$b \leq a \ \wedge \ c \leq a \ \Rightarrow \ b \sqcup c \leq a$$
$$a \sqcap (b \sqcup c) = a \sqcap b \sqcup a \sqcap c$$
$$a \times (b \sqcup c) = a \times b \sqcup a \times c$$

$$a \leq a^* \qquad (a^*)^* \leq a^*$$
$$a^* \times a^* \leq a^*$$
$$(a \sqcup b)^* = a^* \times b^*$$
$$(a \sqcap b)^* = a^* \sqcap b^*$$
$$(a \times b)^* = 1 \sqcup a \times a^* \times b \times b^*$$
$$1^* = 1$$

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The notion of combinatorial reducibility can be adapted to show that deciding validity of an equation is Π_3^p-complete

$\sqcup, 0, 1, \star, \sqcap$ and $(-)^\diamond$ (work in progress)

Composition, iterated composition

Composition $P \star Q$

- Implicitly: ability to make an oracle call to Q then P
- Explicitly: given an instance i of Q and a function that takes a solution of i to an instance of P, compute all relevant solutions

$$\mathsf{dom}(P \star Q) \cong \sum_{i \in \mathsf{dom}(Q)} (Q_i \to \mathsf{dom}(P)) \qquad (P \star Q)_{i,f} \cong \sum_{r \in Q_i} P_{f(r)}$$

Iterated composition $P^\diamond$

- Explicitly: computed as the least fixpoint of $X \mapsto 1 \sqcup (X \star P)$
- Implicitly: ability to make a finite but not fixed in advance number of oracle calls to P

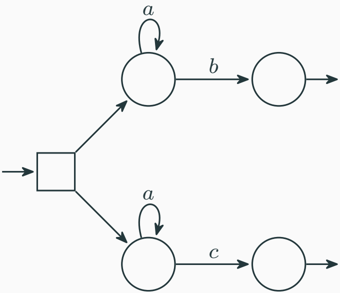
Terms with composition and automata

Starting observation

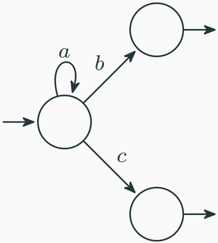
Terms over $0, 1, \sqcup, \star, (-)^\diamond$ = can be regarded as regular expressions.

(alphabet = the set of variables)

- Terms can be mapped to NFAs in a meaningful way
- Adding $\sqcap$ = allowing alternating automata



$(b \star a^\diamond) \sqcap (c \star a^\diamond)$



$(b \sqcup c) \star a^\diamond$

Universal validity and games

Given alternating automata $\mathcal{A}$ and $\mathcal{B}$, we can define a game $\eth(\mathcal{A}, \mathcal{B})$ that captures a notion of simulation such that

Conjecture

$(\mathfrak{W}, 1, 0, \sqcup, \sqcap, \star, (-)^{\diamond}) \models t \leq u$ iff Duplicator wins in $\eth(\mathcal{A}_t, \mathcal{A}_u)$.

Some properties of $\eth(\mathcal{A}, \mathcal{B})$:

- this is a Büchi game
- allows to make several attempts at simulating $\mathcal{A}$ in parallel

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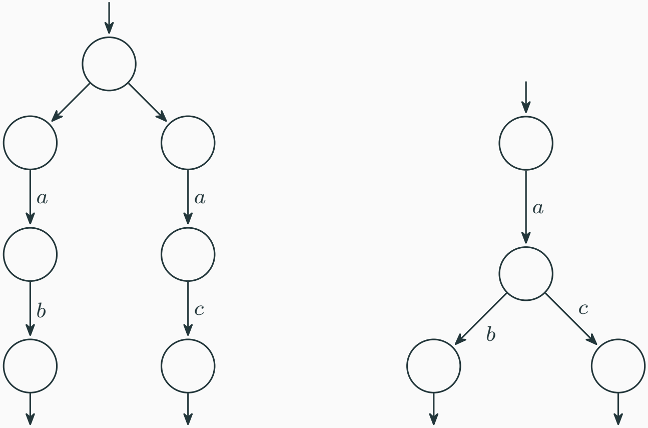
~~Corollary~~ Conjecture

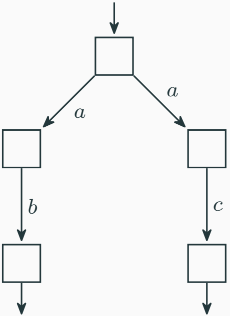
Deciding "$(\mathfrak{W}, 1, 0, \sqcup, \sqcap, \star, (-)^{\diamond}) \models t \leq u$?" is PSPACE-complete.

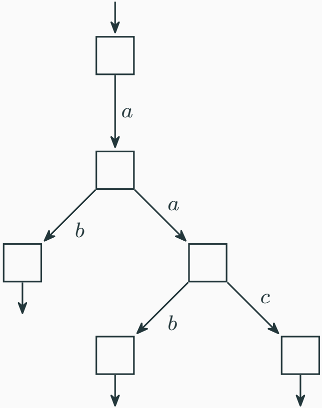
A simple example of simulation and non-simulation



$(b \star a) \sqcup (c \star a) \quad < \quad (b \sqcup c) \star a$

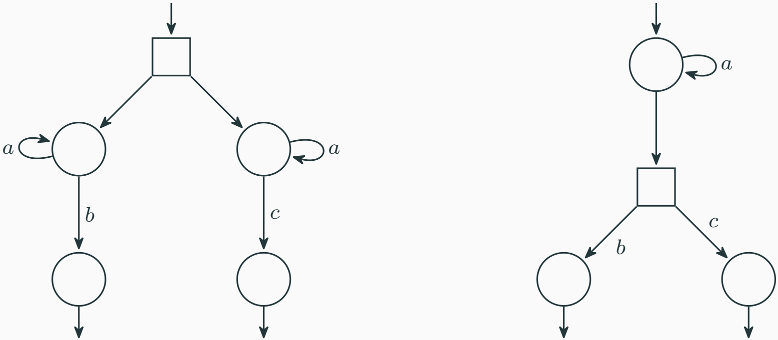
A simulation requiring several concurrent attempts





$$(b \star a) \sqcap (c \star a) \quad < \quad (((c \sqcap b) \star a) \sqcap b) \star a$$

Another simulation requiring several concurrent attempts



$$(b \star a^{\diamond}) \sqcap (c \star a^{\diamond}) \quad \equiv \quad (b \sqcap c) \star a^{\diamond}$$

Induction principles for $(-)^\diamond$

Non-trivial useful axiom for fixpoints (Westrick, 2021)

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$$(x \star y) \sqcap z \leq x \qquad \Rightarrow \qquad (x \star y^\diamond) \sqcap z \leq x$$

(key example: $a^\diamond \sqcap b^\diamond \leq (a \sqcap b)^\diamond$)

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The above axiomatization is not only sound but also complete.

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(x \star y) \sqcap (x \star z) & \leq & x \star (y \sqcap z) \\
(x \star y) \sqcap z & \leq & (x \sqcap z) \star y \\
(x \star y) \sqcap z \leq x & \Rightarrow & (x \star y^\diamond) \sqcap z \leq x
\end{array}
$$

Conjecture

The above axiomatization is not only sound but also complete.

Proof idea: $\exists$ positional simulation strategies, induction on the syntax

Thoughts for further work

- Understand $(\mathfrak{W}, \sqcap, \times, 1)$!!
- What about the Horn theories?
- Establish a similar connection with higher-dimensional automata/concurrency to handle $\sqcup$, $\times$ and $\star$ together?
- Handle the substructure of finitary/first-order/type 1 degrees?
- Can we generalize the proofs turning universal validity into combinatorial characterization for any set of "nice" operations?

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Thanks for listening! Questions?