Quotients of Weihrauch degrees

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CCA 2024, Swansea

Outline

Three "and"s

- $h \leq_W f \sqcup g$, $h \leq_W f \times g$ and $h \leq_W f \star g$ can all be read as "If f and g, then h.".
- Here $f \sqcup g$ receives either a query to f or a query to g as input, and provides a corresponding answer.
- While $f \times g$ receives a query to f and a query to g, and answers both.
- And $f \star g$ receives a query to g, and a way to compute a query to f given any corresponding answer to g, and answers both.

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RT^2_2, SRT^2_2 and COH

- $\mathrm{RT}^2_2 \Leftrightarrow (\mathrm{SRT}^2_2 \wedge \mathrm{COH})$
- Brattka asked how this appears in the Weihrauch degrees.
- $\mathrm{SRT}^2_2 \sqcup \mathrm{COH} <_{\mathrm{W}} \mathrm{RT}^2_2 <_{\mathrm{W}} \mathrm{SRT}^2_2 \star \mathrm{COH}$
- $\left(\mathrm{SRT}^2_2 \times \mathrm{COH}\right) \;|_{\mathrm{W}}\; \mathrm{RT}^2_2$

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & P)

$\mathrm{LPO} \times \mathrm{NON} \not\leq_W \mathrm{RT}^2_2$

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A tighter result

- D^2_2 is "Given a Δ^0_2-subset $A \subseteq \mathbb{N}$, find an infinite set I such that either $I \subseteq A$ or $I \subseteq \mathbb{N} \setminus A$.
- $\mathrm{CFI}_{\Delta^0_2}$ is "Given a cofinite Δ^0_2-subset of $\mathbb{N}$, find an infinite subset of it".

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & P)

$\mathrm{CFI}_{\Delta^0_2} \equiv_W \max_{\leq_W}\{h \mid h \times \mathrm{LPO} \leq_W \mathrm{D}^2_2\}$

Question (Dzhafarov, Goh, Hirschfeldt, Patey & P)

When does $\max_{\leq_W}\{h \mid h \times f \leq_W g\}$ *exist?*

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Returning to the problem

Theorem (Goh, P & Valenti)

$\text{lim} \equiv_W \max_{\leq_W}\{h \mid h \times \widehat{\text{ACC}_{\mathbb{N}}} \leq_W \text{DS}\}$

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Outline

Definition

A residuated lattice is a lattice equipped with a monoidal operation $\cdot$ such that $\max\{h \mid f \cdot h \leq g\}$ and $\max\{h \mid h \cdot f \leq g\}$ exist.

- By asking about the dual of the lattice, also $\min\{h \mid f \leq g \cdot h\}$ and $\min\{h \mid f \leq h \cdot f\}$ are relevant.
- If $\cdot$ is the meet or join, we have a Heyting or Brouwer algebra.
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Residuals, systematic

1. $\max_{\leq_W}\{h \mid f \sqcup h \leq_W g\}$ is boring, it is 0 if $f \leq_W g$ and g otherwise.
2. $\min_{\leq_W}\{h \mid f \leq_W g \sqcup h\}$ does not exist (Higuchi & P)
3. $\max_{\leq_W}\{h \mid f \times h \leq_W g\}$ is our main focus here.
4. $\min_{\leq_W}\{h \mid f \leq_W g \times h\}$ does not exist (Higuchi & P)
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8. $\min_{\leq_W}\{h \mid f \leq_W h \star g\}$ does not exist (Brattka & P).
9. $\max_{\leq_W}\{h \mid f \sqcap h \leq_W g\}$ does not exist (Higuchi & P).
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Constructing the quotient

Definition (Goh, P & Valenti)

Given $F, G :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ with G different from 0, we define their *parallel quotient* $F/G :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as follows:

$$\mathrm{dom}(F/G) := \{\langle n, k, p\rangle \mid \forall q \in \mathrm{dom}(G)\ \Phi_n(\langle p, q\rangle) \in \mathrm{dom}(F) \wedge$$

$$\ldots \forall r \in F(\Phi_n(\langle p, q\rangle))\ \Phi_k(\langle p, q, r\rangle) \in G(q)\}$$

$$F/G(\langle n, k, p\rangle) := \{\langle q, r\rangle \mid q \in \mathrm{dom}(G)\ \wedge\ r \in F(\Phi_n(\langle p, q\rangle))\}$$

Algebraic properties

- If g is pointed, $f/g \leq_{\mathrm{W}} f$
- f/g is pointed iff $g \leq_{\mathrm{W}} f$
- $(f/g)/h \equiv_{\mathrm{W}} f/(g \times h)$
- $(f \sqcap g)/h \equiv_{\mathrm{W}} (f/h) \sqcap (g/h)$
- $f/(g \sqcup h) \equiv_{\mathrm{W}} (f/g) \sqcap (f/h)$

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More algebraic properties

- $(a \sqcup b)/(?c) \equiv_{\mathrm{W}} a/(?c) \sqcup b/(?c)$
- $(F/F)^* \equiv_{\mathrm{W}} F/F$
- F^*/G is either 0 or $F^* \times d_A$ for some $A \subseteq \mathbb{N}^{\mathbb{N}}$

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Implication on the Medvedev degrees

Definition

For $A \subseteq \mathbb{N}^{\mathbb{N}}$, let $d_A : A \to \{0\}$ be the unique such morphism.

Proposition (Higuchi P)

$A \mapsto d_A$ is a lattice embedding of the dual of the Medvedev degrees $\mathfrak{M}^{op}$ into the Weihrauch degrees $\mathfrak{W}$.

Proposition

$d_A / d_B \equiv_W d_{B \to A}$

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Outline

How we actually got to the question

A more systematic alternative history

Constructing quotients and some algebraic properties

Calculating some quotients

Finite closed choice

Proposition

$\mathrm{C}_n / \mathrm{C}_k \equiv_W \mathrm{C}_{\lfloor \frac{n}{k} \rfloor}$

Corollary

$\mathrm{C}_3 / \mathrm{C}_2 \equiv_W 1$

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More on the pigeon hole principle

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$\mathrm{RT}^1_{n+1}/\mathrm{ACC}_{\mathbb{N}} \equiv_W \mathrm{RT}^1_n$

Corollary

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One more

Proposition

$\mathrm{CC}_{[0,1]}/\mathbb{N} \equiv_W \mathrm{C}_2^*$

The end (for now)

That’s all, folks!