New definitions in the theory of Type 1 computable topological spaces

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CCA 2024, Swansea

Outline

Goal

- Motivation: the "intrinsic topology" approach (Escardò, Schröder, Taylor) does not work for Type 1 computability.

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- General definition of a **Type 1 computable topological space**.

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- General definition of a **Type 1 computable topological space**.
- General definition of **the computable topology generated by a computable metric** that does not rely on effective separability.

Outline

Polish spaces

Definition

A **Polish space** is a topological space whose topology comes from a complete metric, and which is separable.

Computable presentation of a Polish space

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If X is a Polish space with a distance d, **a computable presentation** of (X, d) is given by:

- a dense sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$,
- a program which, given n, m, q, produces a rational approximation of $d(x_n, x_m)$ with error strictly less than 2^{-q}.

Representations

Definition

A **representation** of a set X is a partial surjection $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

Computably Polish spaces

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A represented space (X, ρ) is **computably Polish** if:

- it has a **computable metric**,
- which is **computably complete**,
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- and it is **computably separable**.
  - There is a dense and ρ-computable sequence.

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Lemma

A represented space (X, ρ) is computably Polish if and only if some computable presentation induces ρ as a representation.

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Two different approaches to computable Polish spaces:

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  - Better if you already know the represented space you want to study.

The two approaches agree.
If (for some reason) you need the Type 1 definition of a computable Polish space, you can use either of the approaches, and replace representations with numberings.

Outline

Presentation of a topological space via Lacombe basis

Definition (Weihrauch-Grubba, 2009)

A **computable presentation of a topological space** X is a numbered basis $(B_n)_{n\in\mathbb{N}}$ for which there exists a program that, given i and j, produces the code of a c.e. set $I \subseteq \mathbb{N}$ such that

$$B_i \cap B_j = \bigcup_{k\in I} B_k.$$

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See also: Bauer 2000, Lacombe 1957.

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This is not a general notion of a computable topological space, we will see why more precisely.

In my opinion, this is the correct notion of "computably second countable computable topological space".

Representation approach

Given any represented space (X, ρ), there is a representation of the open sets of X called the **Sierpinski representation**.

Sierpinski representation

The **Sierpinski space** $\mathbb{S}$ is $\{\top, \bot\}$ with topology generated by $\{\top\}$. The representation $\tau_{\mathbb{S}}$ of the Sierpinski space is given by

$$\tau_{\mathbb{S}}(0^\omega) = \bot;$$

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The **Sierpinski representation of open sets associated to** ρ is the representation of continuous maps from X to $\mathbb{S}$.

Finite intersections and countable unions are computable for $\rho_{\mathbb{S}}$.

Type 2 computable topological space

We could say that a “Type 2 computable topological space” is a represented space equipped with the final topology of the representation.

Realizability

Bauer, PhD thesis, 2000

Defines, in the general context of realizability:

- The **standard dominance Σ** is given by:
  - $o = \lambda n \in \mathbb{N}.0$;
  - $\sim$ is an equivalence relation on $2^{\mathbb{N}}$ defined by $f \sim g \iff (f = o \iff g = o)$.
  - Σ is the quotient $2^{\mathbb{N}} / \sim$.

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- The **intrinsic topology** on a set X is defined by maps to the standard dominance: $\mathcal{O}(X) = \Sigma^X$.

Fact

Every map is continuous with respect to the intrinsic topology.

In realizability

There are many topologies other than the intrinsic topology!

In Type 2 computable analysis

When we interpret the standard dominance in terms of represented spaces, we get the Sierpinski space, with its usual representation.

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In Type 2 computable analysis

In computable analysis, the only effective topology we work with is the intrinsic topology.

We restrict our attention to intrinsic topologies.
But it works very well, because in practice, when we want to work with a classical topology, we can find a representation that has this topology as intrinsic topology.
And there are many benefits that are gained by working only with intrinsic topologies.

Summary: computable topological spaces

The situation is now summed up as follows:

- The presentation approach is not general enough for my purpose (see later).

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- The presentation approach is not general enough for my purpose (see later).
- The representation approach to computable topologies is not obtained by effectivizing the notion of topological space.

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Type 1 computability

We will use numberings instead of representations: a numbering of X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$.

Type 1 intrinsic topologies

Crucial fact

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See Friedberg's example of a semi-decidable set of computable reals which is not open in the standard topology of the reals.

Mathieu Hoyrup and Christobal Rojas. *On the information carried by programs about the objects they compute*. Theory of Computing Systems, 2016.

Type 1 computable topologies

Conclusion

Defining Type 1 computable topologies, we cannot simply pick up a Type 2 notion or a notion of presentation and adapt it to numberings.

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Type 1 computable topologies

First new idea

To define Type 1 computable topological spaces, we will use the definition of a topological space.

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See also Bauer, *Spreen spaces and the intrinsic KLST theorem,* July 2023 preprint.

Classical definition

Definition

A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a set of subsets of X such that:

1. The empty set and X both belong to $\mathcal{T}$;
2. $\mathcal{T}$ is stable by taking arbitrary unions and finite intersections.

Effective definition

Definition

A *Type 1 computable topological space* is a quadruple $(X, \nu, \mathcal{T}_c, \tau)$ where $\nu :\subseteq \mathbb{N} \to X$ is a numbering of X, $\mathcal{T}_c$ is a subset of $\mathcal{P}(X)$, $\tau :\subseteq \mathbb{N} \to \mathcal{T}_c$ is a numbering of $\mathcal{T}_c$, and such that:

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1. The empty set and X both belong to $\mathcal{T}_c$;
2. The open sets in the image of τ are uniformly ν-semi-decidable;
3. The operations of taking computable unions and finite intersections are computable for τ.

Effective continuity

Definition

A function f between two computable topological spaces $(X, \nu, \mathcal{T}_{1,c}, \tau_1)$ and $(Y, \mu, \mathcal{T}_{2,c}, \tau_2)$ is called *effectively continuous* if the function $f^{-1} : \mathcal{T}_{2,c} \to \mathcal{T}_{1,c}$ is (τ_2, τ_1)-computable.

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Generalize metric spaces without supposing effective separability.

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A *Type 1 computable metric space* is a triple (X, ν, d), where ν is a partial numbering of X and $d : X \times X \to \mathbb{R}$ is a ν-computable metric.
(Given n, m and q, we can compute $d(\nu(n), \nu(m))$ within 2^{-q}.)

Obvious thing to say

The topology associated to a metric space is the topology whose basis are the open balls.

Classical definition: Nogina type

X a set. A basis is a set $\mathcal{B}$ of subsets of X such that:

- Every element of X belongs to an element of $\mathcal{B}$;
- For any two elements B_1 and B_2 of $\mathcal{B}$, and for any x in $B_1 \cap B_2$, there is an element B_3 in $\mathcal{B}$ containing x and such that B_3 is a subset of $B_1 \cap B_2$.

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See Nogina, *Effectively topological spaces*, 1966.

Classical definition: Lacombe type

A basis is a set $\mathcal{B}$ of subsets of X such that:

- The union of elements of $\mathcal{B}$ gives X:

$$\bigcup_{B \in \mathcal{B}} B = X;$$

- The intersection of two elements of the basis can be written as a union of elements of this basis: for any B_1 and B_2 in $\mathcal{B}$, there is a subset $\mathcal{C}$ of $\mathcal{B}$ such that

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A subset O of X is called open if it can be written as a union of basic sets.

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Lacombe bases

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See Lacombe, *Les ensembles récursivement ouverts ou fermés, et leurs applications à l'analyse récursive, 1957*

Problem with Lacombe bases

Lacombe bases do not apply to non-computably separable metric spaces

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In a metric space, we have a function that shows that the open balls form a basis:

$$I(x, B(y, r_1), B(z, r_2)) = \min(r_1 - d(x, y), r_2 - d(x, z)).$$

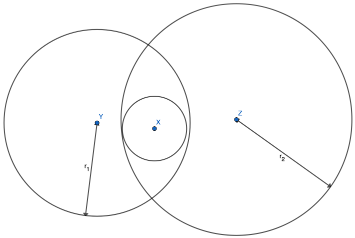

Figure: Computing intersection in a metric space

Moschovakis' Theorem

Theorem (Moschovakis, 1964)

On a Type 1 computable Polish space, the following are equivalent:

- *O is a computable union of open balls with rational radii,*
- *O is a semi-decidable set such that given $x \in O$, we can compute $r \in \mathbb{R}_+$, $B(x, r) \subseteq O$.*

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Theorem (Brattka-Presser, 2003)

In a CMS, the Sierpinski representation of open sets is computably equivalent to the representation where open sets are given by unions of balls.

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Yet another similar theorem is due to Gregoriades, Kispéter, and Pauly, 2016.

Conclusion

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And there are "Moschovakis type theorems" which give sufficient conditions for equality to occur.
Those conditions are different forms of *effective separability* and *effective second countability.*

Left computable reals

Denote by $c_{\nearrow} :\subseteq \mathbb{N} \to \mathbb{R}_{\nearrow}$ the numbering associated to left computable reals.

Description of the Metric topology

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The τ-name of an open set O is an encoded pair (n, m), where n gives the code of O as a semi-decidable set, and where m encodes a $(\nu, c_{\nearrow})$-computable function $F : O \to \mathbb{R}^+_{\nearrow}$, which satisfies the following:

$$\forall x \in O,\ B(x, F(x)) \subseteq O;$$

$$\forall x \in O,\ \exists r > 0,\ \forall y \in B(x, r),\ F(y) > r.$$

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Theorem (R.)

This indeed defines a Type 1 computable topology.

Moschovakis' Theorem on Lacombe sets is trivialized

Theorem

Let (X, ν, d) be a Type 1 computable metric space. If there exists a ν-computable sequence which is dense, then the computably open sets are computable unions of balls, and this is uniform.

Formal inclusion relations

Definition (Spreen, 1998)

Let $\mathfrak{B}$ be a subset of $\mathcal{P}(X)$, and β a numbering of $\mathfrak{B}$. Let $\mathring{\subseteq}$ be a binary relation on $\mathrm{dom}(\beta)$. We say that $\mathring{\subseteq}$ is a *formal inclusion relation for* $(\mathfrak{B}, \beta)$ if the following hold:

- The relation $\mathring{\subseteq}$ is reflexive and transitive (i.e. $\mathring{\subseteq}$ is a preorder);
- $\forall n, m \in \mathrm{dom}(\beta),\ n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m)$.

Je vous remercie pour votre attention

Thank you for your attention