Different Schools of Computable Analysis on the Space of Marked Groups

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Outline

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Goal

- Give a general setting where non-computably Polish spaces naturally arise.
- Explain how Banach-Mazur computability remains relevant in 2024.
- See some natural group properties that have different classifications in terms of classical vs. effective Borel hierarchies.

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Finitely generated groups

Definition

A group is a set of bijections of a set, stable by composition and taking inverses.

A group G is *finitely generated* if there are finitely many elements $S = (g_1, ..., g_k) \in G$ such that any element of G can be expressed as a product of the elements of S and of their inverses:

$$\forall h \in G, \exists n \in \mathbb{N}, \exists (h_1, ..., h_n) \in (S \cup S^{-1})^n, \ h = h_1 h_2 ... h_n$$

So that every element can be seen as a word in $(S \cup S^{-1})^*$.

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Marked Groups

Definition

A *k-marked group* is a finitely generated group G together with a finite generating family $S = (s_1, \dots, s_k)$.

Definition

A *relation* in a marked group (G, S) is a word $R \in (S \cup S^{-1})^*$ which defines the identity: $R = 1$ in the group.

Lemma

A marked group is uniquely determined by the set of relations it satisfies.

Definition

The *word problem* of (G, S) is the set of all its relations. A group has *solvable word problem* when this set is computable.

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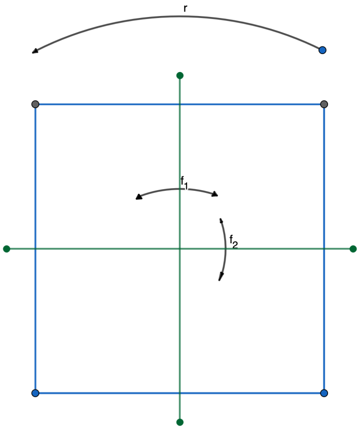

Figure: The dihedral group D_4

Some relations: $f_1^2 = 1,\ r^4 = 1,\ f_2 = f_1 r^2 \ldots$

Consider S fixed, and an enumeration $(R_0, R_1, R_2...)$ of $(S \cup S^{-1})^*$.

A marked group (G, S) is given by a *binary expansion* $(u_n) \in \{0,1\}^{\mathbb{N}}$:

$$u_n = 1 \iff R_n = 1 \text{ in } G$$

Definition

Define a representation $\rho^S_{WP} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathcal{G}$ by mapping the binary expansion of a marked group to the marked group it defines.

Not every element of $\{0,1\}^{\mathbb{N}}$ defines a marked group!

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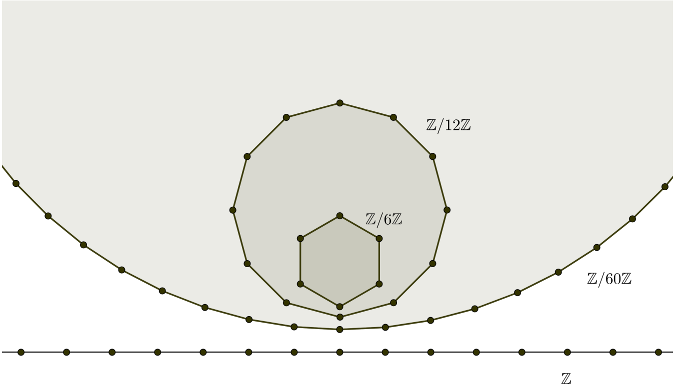
A converging sequence in the space of marked groups

Definition

The final topology of this representation defines what is known as the *space of marked groups.*



Implication between relations

Which elements of $\{0,1\}^{\mathbb{N}}$ define a marked group?

Take $S = (s_1, ..., s_n)$ a set of symbols, $R_1, ..., R_n, U \in (S \cup S^{-1})^*$ words.

Say that

$$R_1 = 1, ..., R_n = 1 \implies U = 1$$

if in any marked group (G, S) where $R_1 = 1$, $R_2 = 1$,..., $R_n = 1$, then also $U = 1$.

Lemma

A sequence $(u_n) \in \{0,1\}^{\mathbb{N}}$ defines a marked group if and only if it is coherent with respect to the above implications.

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Example

$$ab = ba \implies a^2b = ba^2$$

Is it true that

$$ab = ba \not\Longleftarrow a^2b = ba^2?$$

Yes, see the dihedral group before! f_1 and r do not commute, yet $f_1^2 = 1$.

We are describing parts of the first order universal theory of groups.

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Finite Presentations of groups

The Dihedral group D_4 is given by the finite presentation

$$D_4 = \langle f_1, r \mid f_1^2 = 1,\ (rf_1)^2 = 1, r^4 = 1 \rangle$$

These are not *all* the relations satisfied by D_4, but these are *sufficient*, in the sense that the relations implied by those three relations are exactly all the relations satisfied by D_4.

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Finitely presented groups form a discrete set

Theorem (Tietze, Mostowski)

Equality is semi-decidable for marked groups given by finite presentations.

Boone-Novikov

Theorem (Boone-Novikov, 1958)

There exists a finitely presented group (G, S) with undecidable word problem.

Corollary

Implication between relations is semi-decidable but not co-semi-decidable.

Corollary

The set of elements of $\{0,1\}^{\mathbb{N}}$ that define a marked group is co-c.e. closed but not c.e. closed.

In particular it is not *computably separable.*

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Aside: Gödel numberings

Take $\mathcal{L}$ a language over a signature σ, and a total Gödel numbering ν of all the well formed statements in this language.
Each **complete and consistent theory** can then be seen as an element of the Cantor space, which indicates in position n whether or not the nth statement is true in this theory.
For rich enough languages, consistency of finite sets of statements is co-semi-decidable but not decidable.
We get a co-c.e. closed subset of $\{0, 1\}^{\mathbb{N}}$ that is not c.e. closed.

See Emmanuel Jeandel, Enumeration reducibility in closure spaces with applications to logic and algebra, LICS 2017.

Gives many examples of represented spaces where we may want to be be able to do effective descriptive set theory, but to which the classical theory does not apply.

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What happened?

What happened that made researchers study descriptions of groups other than finite presentations?

Groves-Manning-Wilton

Theorem (Groves-Wilton, Manning, 2009)

There exists an algorithm that takes as input a finite presentation of a group and a solution to its word problem and decides whether or not that group is free.

Question (Groves-Manning-Wilton)

What is it that is proved here exactly?

Definition (Groves-Manning-Wilton, 2012)

"Computable modulo the word problem": almost rediscover *Banach-Mazur computability, i.e. the idea that a function is computable if it maps computable sequences to computable sequences.*

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Numberings, Type 1 computability

Definition

A **numbering** is a partial surjection $\nu :\subseteq \mathbb{N} \to X$.

Fix a representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.
Denote X_c **the computable points** of ρ, i.e. the set of points that have a computable name.
Let φ_0, $\varphi_1 \ldots$ be a standard enumeration of partial computable functions on the natural numbers.

Definition

The **derived numbering** associated to ρ is the numbering of X_c given by

$$\nu(i) = x \iff \varphi_i \text{ is total and } \rho(\varphi_i) = x.$$

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Two notions of computable functions

Definition

(X, ν) and (Y, μ) numbered sets.

A function $f : (X, \nu) \to (Y, \mu)$ is **Markov computable** if it there is a computable function $F :\subseteq \mathbb{N} \to \mathbb{N}$ which realizes it.

This function is **Banach-Mazur computable** if it maps ν-computable sequences to μ-computable sequences.

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Why do we insist on Banach-Mazur computability?

P a property of marked groups. $(\mathcal{G}_c, \nu_{WP}^S)$ the derived numbered set coming from $(\mathcal{G}, \rho_{WP}^S)$.

Theorem (R.)

The following are equivalent:

- *There exists a finitely presented group G with solvable word problem where the problem of determining if a finitely generated subgroup of G has P is not semi-decidable;*
- *P is not Banach-Mazur semi-decidable for ν_{WP}^S.*

Equivalence of a local problem (the input of this problem consists in elements of a group) and of a global problem (the input of this problem consists in whole groups).

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First implication

Fix a marked group (G, S) with solvable word problem.
We consider the problem: given elements g_1, g_2, ..., g_n of (G, S), does the group they generate have P?
Given the words over $(S \cup S^{-1})^*$ that define g_1, ..., g_n, we can recover the **code** of an algorithm that solves the word problem in this subgroup.
It is obtained as a composition of the algorithm for (G, S) with an algorithm that translates words over $g_1, ..., g_n$ as words of $(S \cup S^{-1})^*$.
Thus the set of all finitely generated subgroups of (G, S) is a ν_{WP}^S-computable sequence.
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We consider the problem: given elements g_1, g_2, ..., g_n of (G, S), does the group they generate have P?
Given the words over $(S \cup S^{-1})^*$ that define g_1, ..., g_n, we can recover the **code** of an algorithm that solves the word problem in this subgroup.
It is obtained as a composition of the algorithm for (G, S) with an algorithm that translates words over $g_1, ..., g_n$ as words of $(S \cup S^{-1})^*$.
Thus the set of all finitely generated subgroups of (G, S) is a ν_{WP}^S-computable sequence.
Thus if P is Banach-Mazur semi-decidable, we can semi-decide it for the subgroups of G.

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Reverse implication: Higman's Embedding Theorem

Theorem (Higman 1961, Clapham 1967, Valiev 1975)

Any finitely generated group with solvable word problem embeds into a finitely presented group with solvable word problem.

Explaining old results

Different constructions of finitely presented groups are now understood as being Banach-Mazur results in disguise, embedded inside finitely presented groups.

Examples

- McCool, Unsolvable problems in groups with solvable word problem. *Canadian Journal of Mathematics*, 1970.
- Lockhart, Decision problems in classes of group presentations with uniformly solvable word problem. *Archiv der Mathematik*, 1981
- Duda and Ivanov, On decidability of amenability in computable groups. *Archive for Mathematical Logic*, 2022.

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Conclusion

On the represented space $(\mathcal{G}, \rho^S_{WP})$:

- Whenever proving that something is computable, we want to do it using Type 2 computability.
- Whenever proving something non-computable, we want to do it using Banach-Mazur computability.

In each case, that's the stronger result.

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Outline

Descriptive set theory on the space of marked groups

Classifying properties in the Borel hierarchy on the space of marked groups is an important and difficult topic in the study of discrete countable groups.

Being abelian is clopen

A group G is abelian if

$$\forall a, b \in G,\ ab = ba.$$

Thus this is a closed property.
In fact, it is even clopen, because all elements in G commute if and only if the generators of G commute.

This is a very trivial example of a "local-to-global" phenomenon, but there are non-trivial ones.
Write $aba^{-1}b^{-1} = [a, b]$. Then:

$$\forall a, b, c \in G,\ [a, b]c = c[a, b] \longrightarrow \text{Clopen;}$$

$$\forall a, b, c, d \in G,\ [a, b][c, d] = [c, d][a, b] \longrightarrow \text{Closed, not open;}$$

$$\forall a \in G,\ a^5 = 1 \longrightarrow \text{Unknown.}$$

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Three classifications

There are three classifications we want to obtain:

- In the Classical Borel Hierarchy,
- In the Effective Borel Hierarchy that comes from the representation,
  - Note that classical effective descriptive set theory does not apply.
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Markov vs Type 2 computability

(X, ρ) and (Y, τ) represented spaces, (X_c, ν) and (Y_c, μ) derived numbered sets.

$f : X \to Y$ is (ρ, τ)-computable $\implies f : X_c \to Y_c$ is (ν, μ)-computable.

- So it should be harder to prove that something is Type 2 computable than to prove it Type 1 computable... but in practice it isn't.

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Continuity theorems

There are *continuity theorems* that provide sufficient conditions for Type 1 computable functions to be Type 2 computable, which would allow us to transfer automatically topological information into information about undecidability of problems.

The most classical requirement is the considered space to be a computable Polish space.

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Markov's Lemma

A property P is called **effectively sequentially not open** if there is a computable sequence of elements without P that converges towards an element with P.

Theorem (Markov, 1954)

In a recursive metric space with an algorithm that computes limits of fast Cauchy sequences, we have for a property P:

$$P \text{ effectively sequentially not open} \implies P \text{ not BM semi-decidable}$$

To contrast with:

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Classification

Clopen/decidable properties
Being abelian (all elements commute);
Being isomorphic to a given finite group;
Having cardinality at most n, $n \in \mathbb{N}^*$;
Being Nilpotent of derived length $k > 0$.

Classification

Open/semi-decidable properties
Being nilpotent;
Kazhdan's Property (T);
Having a non-trivial center;
Being perfect;
Having Torsion;
Being virtually cyclic;
Having polynomial growth;
Being a counterexample to Kaplansky's conjecture;
Being polycyclic.

Classification

Closed/co-semi-decidable properties
Being infinite;
Being k-solvable, for $k > 1$;
Having a finite exponent;
Being a limit group;
Being Orderable;
Being δ-hyperbolic, $\delta > 0$;
Having Infinite Conjugacy Classes (ICC).

Classification

Neither closed nor open properties
Being solvable;
Being amenable;
Being simple;
Having sub-exponential growth;
Being finitely presented;
Being hyperbolic;
Being residually finite.

On the classification

In each case, we have the strongest possible statement:

- Open properties are Type 2 c.e. open;
- Non-open properties are even effectively sequentially non-open, and thus not Banach-Mazur semi-decidable.

What does not fit into the correspondence

Remarkably, there are natural properties whose topological classification does not match their classifications in the effective hierarchies.

Some results -many conjectures!

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Isolated groups

Definition

An **isolated group** is an isolated point in the topology of the space of marked groups.
This simply means that there is a finite number of relations and non-relations that this group is the only one to satisfy.

Analogy with logic

Isolated groups correspond to **finitely axiomatizable complete theories**.

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The set of isolated groups, while open, is not c.e. open, nor Banach-Mazur semi-decidable.

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Theorem (Slobodskoi, 1981)

The universal theory of finite groups is undecidable.

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The set of LEF groups is not co-c.e. closed.

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Is the set of LEF groups with solvable word problem Banach-Mazur co-semi-decidable?

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Our new implication rule is co-semi-decidable and not decidable.

Implication between relations in LEF is given by "is it the case that in every **finite** group where $R_1 = 1, ..., R_n = 1$ hold, $U = 1$ is also true?".

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Summary

$$\mathrm{LEF} \subseteq \mathcal{G} \subseteq \{0,1\}^{\mathbb{N}},$$

- $\mathcal{G}$ is co-c.e. closed but not c.e. closed in the Cantor Space, and thus $(\mathcal{G}, \rho^S_{WP})$ is not computably polish, but computably compact;
- LEF is c.e. closed (by definition) but not co-c.e. closed in the Cantor Space, and thus $(\mathrm{LEF}, \rho^S_{WP})$ is computably Polish and compact but not computably compact.

Continuity problem

Problem

Is it true that Banach-Mazur computable functions defined on $\mathcal{G}_c$ are continuous?

We would have

$$P \text{ not clopen} \implies P \text{ not Banach-Mazur decidable.}$$

See

- Peter Hertling, Banach–Mazur Computable Functions on Metric Spaces, CCA 2000
- Dieter Spreen, Some results related to the continuity problem. *Mathematical Structures in Computer Science*, 2016
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