Chains and antichains in the Weihrauch degrees

Manlio Valenti

manlio.valenti@swansea.ac.uk

jww with Steffen Lempp and Alberto Marcone

Swansea University

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Weihrauch reducibility

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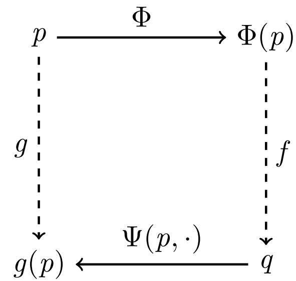

- Given $p \in \mathrm{dom}(g)$, $\Phi(p) \in \mathrm{dom}(f)$
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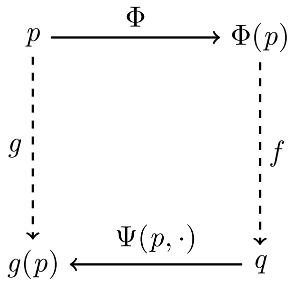

More general spaces can be considered, but problems on $\mathbb{N}^\mathbb{N}$ are enough to study Weihrauch degrees.

The structure of Weihrauch degrees

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The existence of a "natural" top element is equivalent to a (relatively weak) form of choice.

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Warning: the operations $\bigsqcup_{n\in\mathbb{N}} f_n$ and $\bigsqcap_{n\in\mathbb{N}} f_n$ are not degree-theoretic!

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Theorem (Lempp, Miller, Pauly, Soskova, V.)

The Weihrauch degrees above id are dense.

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Muchnik reducibility: non-uniform version of $\le_{\mathrm{M}}$

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Medvedev vs Weihrauch

"The first half of a Weihrauch reduction is a Medvedev reduction".

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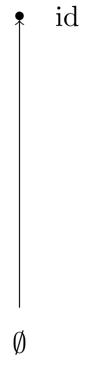
It follows that: $B \le_{\mathrm{M}} A$ iff $d_A \le_{\mathrm{W}} d_B$

This embedding reverses the Medvedev order!

Weihrauch degrees



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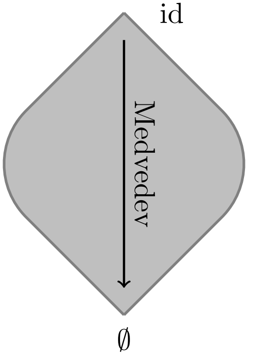

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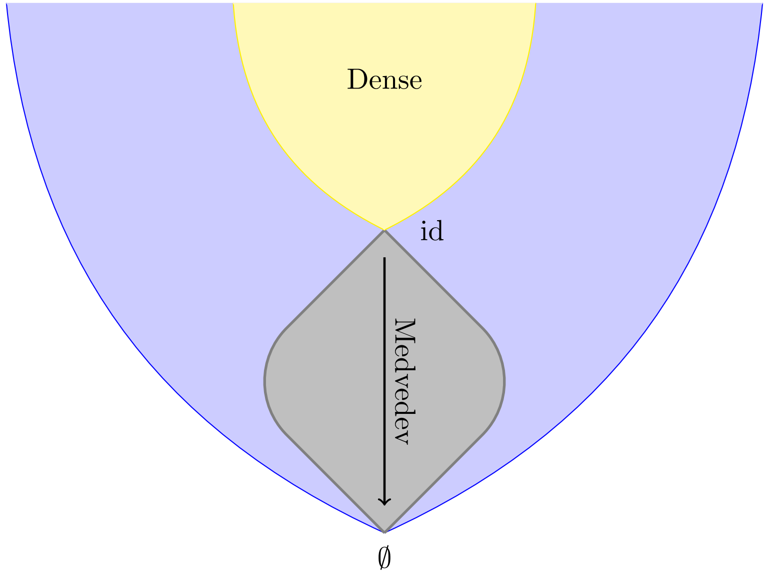

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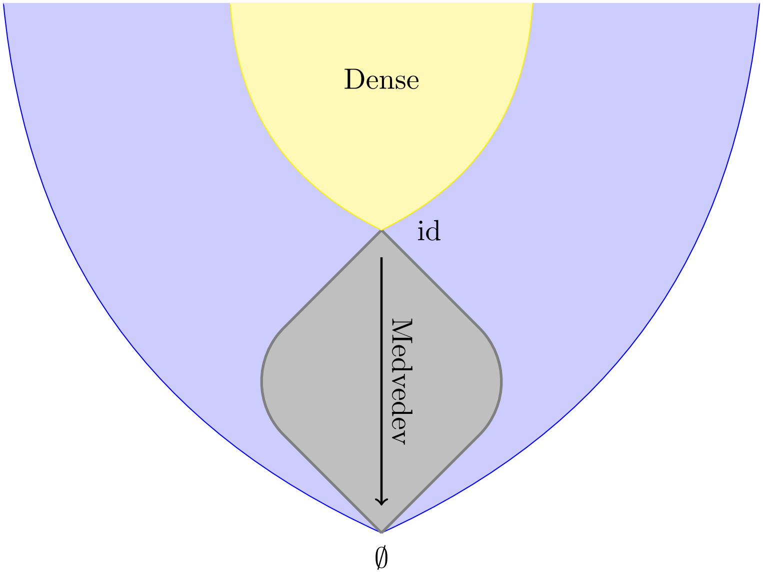

Some results on the structure of Weihrauch degrees are obtained as corollaries of structural results on the Medvedev lattice.

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Is this peculiar of ω_1?

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Let κ be a cardinal with $\mathrm{cof}(\kappa) > \omega$, and let $(f_\alpha)_{\alpha<\kappa}$ be a chain of order type κ. TFAE:

1. $(f_\alpha)_{\alpha<\kappa}$ has an upper bound in $\mathcal{W}$;
2. there is $E \subseteq \kappa$ with $|E| = \kappa$ s.t. for every $p \in \bigcup_{\alpha\in E} \mathrm{dom}(f_\alpha)$, $\bigcap_{\alpha\in I_p^E} f_\alpha(p) \neq \emptyset$

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Let $\mathcal{T}$ denote the Turing degrees. The following are equivalent:

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The same result holds for the Medvedev degrees.

Theorem (Lempp, Marcone, V.)

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1. CH;
2. There is a cofinal chain in $\mathcal{M}_0$ (of order type ω_1).

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Since $k = \mathrm{cof}(k) > \omega_1$,

$$\sup\{\alpha_\beta : \beta < \omega_1\} = \gamma < k,$$

hence f_γ is an upper bound for $(g_\beta)_{\beta<\omega_1}$. Contradiction.

Antichains in $\mathcal{M}$ and $\mathcal{W}$

Proposition (essentially Sorbi, Platek)

There are maximal antichains in $\mathcal{M}$ of size κ, for every $1 \le \kappa \le \mathfrak{c}$ or $\kappa = 2^{\mathfrak{c}}$.

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This does not generalize to Weihrauch degrees!

Proposition (Dzhafarov, Lerman, Patey, Solomon)

For every countable family $\{f_n\}_{n\in\mathbb{N}}$ of non-trivial problems there is g s.t. for every n, $g \mid_{\mathrm{W}} f_n$. In particular, every countable antichain is extendible.

Maximal antichains

The result by (DLPS) cannot be extended to $\mathfrak{c}$-sized families:

Proposition (Lempp, Marcone, V.)

There is a family $\{f_p\}_{p\in\mathbb{N}^\mathbb{N}}$ of non-trivial problems s.t. $g \neq \emptyset$ there is $p \in \mathbb{N}^\mathbb{N}$ s.t. $f_p \leq_W g$.

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For the other direction, we define g so that $g((e,i)^\frown p)$ witnesses that $g \not\le_W f_p$ via Φ_e, Φ_i.

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Since the set-cofinality of $\mathcal{M}_0$ is $\mathfrak{c}$:

Corollary (Lempp, Marcone, V.)

No antichain $\{f_\alpha\}_{\alpha<\kappa}$ in $\mathcal{W}$ with $\kappa < \mathfrak{c}$ is maximal.

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Are there maximal antichains of size $\mathfrak{c}$ in $\mathcal{W}$?

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Minimal degrees and minimal covers

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Theorem (Lempp, Miller, Pauly, Soskova, V.)

The following are equivalent:

1. f is a strong minimal cover of h in the Weihrauch degrees.
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The following are equivalent for a Weihrauch degree g:

1. $\mathrm{id} \not\le_{\mathrm{W}} g$
2. There are $g \le_{\mathrm{W}} h <_{\mathrm{W}} f$ such that (h, f) is an empty interval.

Weihrauch degrees