A note on making analytic sets open[1]

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Introduction

- A subset A of a quasi-Polish space X is Σ^1_1 (lightface analytic) iff $A = f(Z)$ for some computable $f\colon Z \to X$ for some (lightface) Π^0_2-set $Z \subseteq \mathbb{N}^{\mathbb{N}}$.
- The Gandy-Harrington space is the space obtained by refining the topology on $\mathbb{N}^{\mathbb{N}}$ by adding all Σ^1_1-sets as open sets.
- Although the Gandy-Harrington space is not (quasi-)Polish, it has a useful completeness property (it is a strong Choquet space), and has many applications in effective descriptive set theory.
- In this talk, we present a generalized construction for effective quasi-Polish spaces.

- Let X be an effective quasi-Polish space, and $\widehat{X}$ the refinement of X that adds each (lightface) Σ^1_1-subset as a c.e.-open subset.
- Although $\widehat{X}$ is not quasi-Polish in general, there is a "nice" computable embedding of $\widehat{X}$ into an effective quasi-Polish space Y.

Theorem

There is a computable procedure which converts a code for an effective quasi-Polish space X into a code for an effective quasi-Polish space Y and a code for a computable retraction $f\colon Y \to X$ such that:

- $f^{-1}(\{x\})$ has a unique maximal element for each $x \in X$
- the subspace $M \subseteq Y$ of maximal elements of the fibers of f is computably homeomorphic to $\widehat{X}$
- the restriction of f to M corresponds to the canonical map from $\widehat{X}$ to X

- In particular, Gandy-Harrington space computably embeds as the subspace of maximal elements of an effective quasi-Polish space (and similarly for any other T_1-quasi-Polish space).

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Quasi-Polish spaces

- Quasi-Polish spaces are a class of well-behaved countably based T_0-spaces.
  - Quasi-Polish $=$ countably based & completely quasi-metrizable.
- All Polish spaces and all countably based locally compact sober spaces are quasi-Polish.
  - A quasi-Polish space is metrizable if and only if it is Polish.
  - It follows that all ω-continuous domains and all countably based spectral spaces are quasi-Polish.
- The descriptive set theory for Polish spaces naturally generalizes to quasi-Polish spaces.
  - If X is quasi-Polish, then $A \subseteq X$ is quasi-Polish iff $A \in \mathbf{\Pi}^0_2(X)$.
    - A subset S is $\mathbf{\Pi}^0_2$ iff there are sequences $(U_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ of opens such that $x \in S \iff (\forall i \in \mathbb{N})\,[x \in U_i \Rightarrow x \in V_i]$.
  - Extending a quasi-Polish topology with countably many $\mathbf{\Delta}^0_2$-sets results in a quasi-Polish topology.
  - If X is quasi-Polish and $A \subseteq X$ is Borel, then there is a quasi-Polish topology that refines the topology on X and such that A is open in the refinement.

Spaces of Ideals

The following definition is useful for characterizing effective quasi-Polish spaces.

Definition

Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an ideal (with respect to $\prec$) if and only if:

1. $I \neq \emptyset$, *(I is non-empty)*
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, *($I$ is a lower set)*
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c \,\&\, b \prec c)$. *($I$ is directed)*

The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$ for $n \in \mathbb{N}$.

- Think of $\mathbb{N}$ as encoding pieces of information about points in an abstract space, where $a \prec b$ means b contains more information than a.
- Then a point (i.e., an ideal $I \in \mathbf{I}(\prec)$) is any consistent collection of arbitrarily precise information.

Theorem (d., A. Pauly, & M. Schröder, 2019)

A space is quasi-Polish if and only if it is homeomorphic to a space of the form $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.

- We can also use other countable sets (encoded by $\mathbb{N}$)
- Examples:
  1. If $\prec_{\mathbb{N}^\mathbb{N}}$ is the strict prefix relation on the set $\mathbb{N}^{<\mathbb{N}}$ of finite sequences of natural numbers, then $\mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}})$ is homeomorphic to the Baire space $\mathbb{N}^\mathbb{N}$.
  2. If $\subseteq$ is the usual subset relation on the set $\mathcal{P}_{\text{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$, then $\mathbf{I}(\subseteq)$ is homeomorphic to $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.
  3. Let (X, d) be a separable metric space. Fix a countable dense subset $D \subseteq X$. Define the transitive relation $\prec_d$ on $D \times \mathbb{N}$ as

$$\langle x, n \rangle \prec_d \langle y, m \rangle \iff d(x, y) < 2^{-n} - 2^{-m}.$$

$\mathbf{I}(\prec_d)$ is homeomorphic to the metric completion of (X, d).

Effective quasi-Polish spaces

Definition

An effective quasi-Polish space is a represented space computably homeomorphic to $\mathbf{I}(\prec)$ for some c.e. transitive relation $\prec$ on $\mathbb{N}$.

- We sometimes call $\prec$ a code for the space.

This is equivalent to the following:

- "Computable quasi-Polish spaces" (M. Korovina and O. Kudinov, 2017).
- "Precomputable quasi-Polish spaces" (M. d, A. Pauly, & M. Schröder, 2019).
- "Effective quasi-Polish space" (M. d, T. Kihara, & V. Selivanov, 2022).

Definition

An effective quasi-Polish space $\mathbf{I}(\prec)$ is overt if $E_\prec = \{n \in \mathbb{N} \mid [n]_\prec \neq \emptyset\}$ is also c.e.

This is equivalent to:

- "Effectively enumerable computable quasi-Polish spaces" (M. Korovina and O. Kudinov, 2017).
- "Computable quasi-Polish spaces" (M. d, A. Pauly, & M. Schröder, 2019)
- "Effective quasi-Polish spaces" (M. Hoyrup, C. Rojas, V. Selivanov, & D. Stull, 2019)
- "C.e. effective quasi-Polish space" (M. d, T. Kihara, & V. Selivanov, 2022).

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Complete computable topological spaces

Definition

A (countably based) computable topological space is a tuple (X, φ, S) where:

1. X is a T_0-space,
2. $\varphi\colon \mathbb{N} \to \mathbf{O}(X)$ is an enumeration of a basis for X,
3. $S \subseteq \mathbb{N}^3$ is a c.e. set satisfying $\varphi(n) \cap \varphi(m) = \bigcup\{\varphi(k) \mid \langle n, m, k\rangle \in S\}$ for each $n, m \in \mathbb{N}$.

- A computable topological space (X, φ, S) is given the representation $\rho_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ defined as

$$\rho_X(p) = x \iff \{n \mid x \in \varphi(n)\} = \{n \mid (\exists i)\, p(i) = n\}.$$

- If (X, φ, S) is a computable topological space, and $e\colon Y \to X$ is an embedding, then (Y, ψ, S) is also a computable topological space, where $\psi(i) = e^{-1}(\varphi(i))$.

  (ψ just restricts the basic open subsets of X to the subspace Y)

Definition

A computable topological space (X, φ, S) is complete iff for any computable topological space (Y, ψ, S) there is a **unique** computable embedding $e\colon Y \to X$ satisfying $\psi(i) = e^{-1}(\varphi(i))$.

Complete computable topological spaces are unique up to computable isomorphism, and they are precisely the effective quasi-Polish spaces:

Theorem

- If $\prec$ is a c.e. transitive relation on $\mathbb{N}$, then $(\mathbf{I}(\prec), \varphi\colon n \mapsto [n]_{\prec}, \{\langle a, b, c\rangle \mid a \prec c \,\&\, b \prec c\})$ is a complete computable topological space.
- Conversely, for every c.e. subset $S \subseteq \mathbb{N}^3$ there is a complete computable topological space (X, φ, S) where X is an effective quasi-Polish space.

Upper and lower powerspaces

Definition

Given a topological space X with topology $\mathbf{O}(X)$, define the topological spaces $\mathbf{A}(X)$ and $\mathbf{K}(X)$ as follows:

- $\mathbf{A}(X)$ (Lower powerspace):
  - Set of closed subsets of X with lower Vietoris topology, which has subbasis $\Diamond U := \{A \in \mathbf{A}(X) \mid A \cap U \neq \emptyset\}$ for $U \in \mathbf{O}(X)$
- $\mathbf{K}(X)$ (Upper powerspace):
  - Set of saturated compact subsets of X with upper Vietoris topology, which has subbasis $\Box U := \{K \in \mathbf{K}(X) \mid K \subseteq U\}$ for $U \in \mathbf{O}(X)$

Note: $S \subseteq X$ is saturated iff $S = \bigcap\{W \in \mathbf{O}(X) \mid S \subseteq W\}$.
(Every subset of a T_1-space is saturated).

Powerspaces of spaces of ideals

Somewhat surprisingly, several constructions from domain theory (originally due to M. Smyth) still apply in this generality.

Theorem (d.)

Let $\prec$ be a binary transitive relation on $\mathbb{N}$. Define binary transitive relations $\prec_L, \prec_U$ on $\mathcal{P}_{fin}(\mathbb{N})$ as follows:

- $A \prec_L B$ iff $(\forall a \in A)(\exists b \in B)\, a \prec b$,
- $A \prec_U B$ iff $(\forall b \in B)(\exists a \in A)\, a \prec b$.

Then

- $\mathbf{I}(\prec_L) \cong \mathbf{A}(\mathbf{I}(\prec))$, the lower powerspace of $\mathbf{I}(\prec)$ (i.e. the set of closed sets with the lower Vietoris topology)
- $\mathbf{I}(\prec_U) \cong \mathbf{K}(\mathbf{I}(\prec))$, the upper powerspace of $\mathbf{I}(\prec)$ (i.e. the set of saturated compact sets with the upper Vietoris topology)

Effectively extending topologies

- It is well known that if X is a Polish space and $A \subseteq X$ is closed, then the topology generated by adding A as an open set to the topology of X is again Polish.
- More generally, if countably many $\mathbf{\Delta}^0_2$-sets are added to the topology of a quasi-Polish space then the resulting space will be quasi-Polish.
  - Adding non-closed $\mathbf{\Delta}^0_2$-sets to the topology of a Polish space can result in a non-regular (hence non-Polish) quasi-Polish topology.

In recent joint work with T. Kihara and V. Selivanov we proved an effective version of the topology extension result for closed sets.

Theorem (d., T. Kihara, & V. Selivanov, 2023)

Given a c.e. code for an effective quasi-Polish space X and a c.e. sequence $(A_i)_{i\in\mathbb{N}}$ of co-c.e. closed subsets of X, one can effectively obtain a c.e. code for the space obtained by adding each A_i $(i \in \mathbb{N})$ as a c.e. open set to the topology of X.

Open problem

Effectivize the result for $\mathbf{\Delta}^0_2$-sets.

Note: The above Theorem can be used to make Δ^0_2 (and more complicated Borel) sets open, but there are many ways to refine the topology. The open problem is to find the topology *generated* by adding a given Δ^0_2-set (i.e., the *coarsest* refinement).

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Main result

Given an effective quasi-Polish space X, we write $\widehat{X}$ for the space obtained by adding all lightface Σ^1_1-subsets as open sets.

Theorem

There is a computable procedure which converts a code for an effective quasi-Polish space X into a code for an effective quasi-Polish space Y and a code for a computable retraction $f\colon Y \to X$ such that:

- each fiber of f has a unique maximal element (w.r.t. the specialization order of Y)
- the subspace $M \subseteq Y$ of maximal elements of the fibers of f is computably homeomorphic to $\widehat{X}$
- the restriction of f to M corresponds to the canonical map from $\widehat{X}$ to X

Details: Computable functions

Definition

Let $\prec_1$ and $\prec_2$ be c.e. transitive relations on $\mathbb{N}$.

- A code for a partial function is any subset $R \subseteq \mathbb{N} \times \mathbb{N}$.
- Each code R represents the partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ defined as

$$\begin{aligned} \ulcorner R \urcorner(I) &= \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n\rangle \in R\}, \\ dom(\ulcorner R \urcorner) &= \{I \in \mathbf{I}(\prec_1) \mid \ulcorner R \urcorner(I) \in \mathbf{I}(\prec_2)\}. \end{aligned}$$

Theorem

Let $\prec_1$ and $\prec_2$ be c.e. transitive relations on $\mathbb{N}$. A total function $f \colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is computable if and only if there is a c.e. code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

Details: Encoding analytic sets

- Let $\prec$ be a c.e. transitive relation on $\mathbb{N}$.
- Recall that $\prec_{\mathbb{N}^\mathbb{N}}$ is a code for $\mathbb{N}^\mathbb{N}$.
- Let $(R_i)_{i\in\mathbb{N}}$ be an enumeration of all $(\prec_{\mathbb{N}^\mathbb{N}}, \prec)$-closed c.e. partial function codes.
  - R is $(\prec_1, \prec_2)$-closed if for each $\langle m, n\rangle \in R$, if $m \prec_1 m'$ then $\langle m', n\rangle \in R$ and if $n' \prec_2 n$ then $\langle m, n'\rangle \in R$. The $(\prec_1, \prec_2)$-closure of R can be enumerated given R, $\prec_1$, and $\prec_2$, and taking the closure does not change the interpretation of codes for total functions.
  - Using closed codes is not essential, but simplifies some definitions later.
- Define $A_i = \{I \in \mathbf{I}(\prec) \mid (\exists P \in \mathbf{I}(\prec_{\mathbb{N}^\mathbb{N}}))\ulcorner R_i \urcorner(P) = I\}$.
  A_i is the range of the partial function $\ulcorner R_i \urcorner :\subseteq \mathbb{N}^\mathbb{N} \to \mathbf{I}(\prec)$.
  A_i is Σ^1_1 because $\ulcorner R_i \urcorner$ is computable and $dom(R_i)$ is Π^0_2.
- Therefore, $(A_i)_{i\in\mathbb{N}}$ is an effective enumeration of all Σ^1_1-subsets of $\mathbf{I}(\prec)$.

Details: Main construction

- Let $\prec$ and $(R_i)_{i\in\mathbb{N}}$ be as in the previous slide.
  - $R_i^{(n)}$ is the finite subset of R_i enumerated within n steps.
- $\mathcal{P}_{\text{fin}}(S)$ is the set of all finite subsets of S.
- Define the c.e. relation $\sqsubset$ on $\mathbb{N} \times \mathbb{N} \times \mathcal{P}_{\text{fin}}(\mathbb{N}^{<\mathbb{N}} \times \mathbb{N})$ as $\langle m, x, F\rangle \sqsubset \langle n, y, G\rangle$ if and only if the following all hold:
  1. $m < n$, $x \prec y$, and $F \subseteq G$ (monotonicity),
  2. $(\forall\langle\sigma, i\rangle \in F)(\forall\langle\rho, w\rangle \in R_i^{(m)})\,[\rho \prec_{\mathbb{N}^\mathbb{N}} \sigma \implies w \prec y]$, and
  3. $(\forall\langle\sigma, i\rangle \in F)(\exists\langle\tau, i\rangle \in G)\,[\sigma \prec_{\mathbb{N}^\mathbb{N}} \tau \,\&\, \langle\tau, x\rangle \in R_i]$.
- Define $f\colon \mathbf{I}(\sqsubset) \to \mathbf{I}(\prec)$ as

$$f(J) = \{x \in \mathbb{N} \mid (\exists m, F)\,\langle m, x, F\rangle \in J\}$$

and define $g\colon \mathbf{I}(\prec) \to \mathbf{I}(\sqsubset)$ as

$$g(I) = \{\langle m, x, \emptyset\rangle \mid m \in \mathbb{N} \,\&\, x \in I\}.$$

Then f is a computable retract with computable section g.

- Intuitively, $\langle m, x, F \rangle$ contains the information:
  - m is the number of computation steps,
  - x is information about some $I \in \mathbf{I}(\prec)$,
  - $\langle \sigma, i \rangle \in F$ encodes some $\sigma \prec_{\mathbb{N}^{\mathbb{N}}} P \in \mathbb{N}^{\mathbb{N}}$ with $\ulcorner R_i \urcorner(P) = I$.
- A $\sqsubset$-ideal J is encoding a $\prec$-ideal $I = f(J)$ along with possible witnesses showing which $\ulcorner R_i \urcorner$ have I in its image (i.e., which A_i contain I).
  - The unique maximal $\sqsubset$-ideal in $f^{-1}(\{I\})$ contains **all** the information of which $\ulcorner R_i \urcorner$ have I in its image.

Theorem

There is a computable procedure which converts a code for an effective quasi-Polish space X into a code for an effective quasi-Polish space Y and a code for a computable retraction $f\colon Y \to X$ such that:

- each fiber of f has a unique maximal element (w.r.t. the specialization order of Y)
- the subspace $M \subseteq Y$ of maximal elements of the fibers of f is computably homeomorphic to $\widehat{X}$
- the restriction of f to M is the canonical map from $\widehat{X}$ to X

Application 1: Strong Choquet space

Definition

Let X be a space. The strong Choquet game is played as:

Player I:	(x_0, U_0),	(x_1, U_1),	$\ldots$
Player II:	V_0,	V_1,	$\ldots$

with U_i, V_i open and $x_i \in V_i \subseteq U_i \subseteq V_{i-1}$. Player II wins iff $\bigcap_{i\in\mathbb{N}} V_i \neq \emptyset$. X is a strong Choquet space iff Player II has a winning strategy.

It is well known that the Gandy-Harrington space is a strong Choquet space.

Corollary

$\widehat{X}$ is a strong Choquet space for any effective quasi-Polish space X.

Proof: Assume $X \cong \mathbf{I}(\prec)$ and let $\widehat{X} \cong M \subseteq \mathbf{I}(\sqsubset)$ be as in the main result. On round i, if Player I plays $(J_i, [n_i]_\sqsubset)$ with $J_i \in M$, then Player II responds with $[m_i]_\sqsubset$ satisfying:

- $m_i \in J_i$ (so $J_i \in [m_i]_\sqsubset$),
- $n_i \sqsubset m_i$ (so $[m_i]_\sqsubset \subseteq [n_i]_\sqsubset$), and
- $m_{i-1} \sqsubset m_i$ when $i > 0$ (so $(m_i)_{i\in\mathbb{N}}$ is $\sqsubset$-ascending).

Then $(m_i)_{i\in\mathbb{N}}$ generates a $\sqsubset$-ideal J, which is contained in some maximal $J' \in M$. Thus $M \cap \bigcap_{i\in\mathbb{N}}[m_i]_\sqsubset \neq \emptyset$, so this strategy is winning for Player II. □

Note: The strategy in the above proof is computable if it is assumed that Player I provides an enumeration of J_i when $(J_i, [n_i]_\sqsubset)$ is played.

Application 2: Silver's dichotomy

The following generalizes a well known result by Silver for Polish spaces. The proof is essentially the same as Harrington's proof.

Theorem

If E is a $\mathbf{\Pi}^1_1$ equivalence relation on a quasi-Polish space X, then either E has countably many equivalence classes or there is continuous $g\colon 2^{\mathbb{N}} \to X$ such that $p \neq q$ implies $\neg E(g(p), g(q))$.

Conclusion

- Let X be an effective quasi-Polish space, and $\widehat{X}$ the refinement of X that adds each (lightface) Σ^1_1-subset as a c.e.-open subset.

Theorem

There is a computable procedure which converts a code for an effective quasi-Polish space X into a code for an effective quasi-Polish space Y and a code for a computable retraction $f\colon Y \to X$ such that:

- $f^{-1}(\{x\})$ has a unique maximal element for each $x \in X$
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