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Continuity and Computability of Relations

Vasco Brattka
Peter Hertling

FernUniversität
- Gesamthochschule - in Hagen
Fachbereich Informatik
Postfach 940
D-58084 Hagen

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Vasco Brattka
Peter Hertling

Theoretische Informatik I
FernUniversität Hagen
D-58084 Hagen
Germany

e-mail: `vasco.brattka@fernuni-hagen.de`
`peter.hertling@fernuni-hagen.de`

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Abstract

The main theorem of the theory of effectivity (cf. Kreitz and Weihrauch [KW1], [W1]) states that in admissibly represented topological spaces a function is continuous iff it has a continuous representation. Hence continuity is a necessary condition for computability.

We investigate an extended model of computability in order to compute relations. From another point of view these relations are nondeterministic operations or set-valued functions. We show that for a special class of topological spaces (including the complete separable metric ones) and for a certain notion of continuity for relations the main theorem can be extended too.

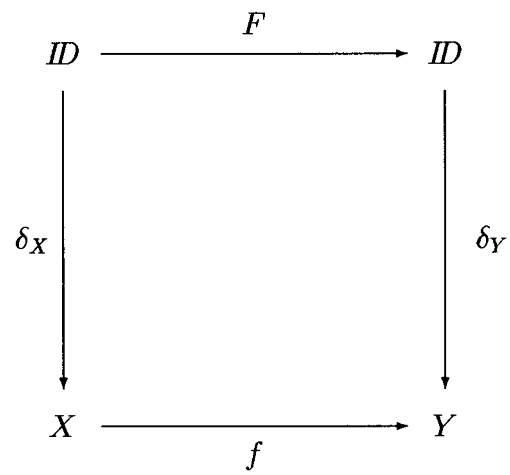
Contents

1 Introduction

The theory of effectivity handles computation in spaces X with the cardinality of the continuum, hence most spaces which are of interest in analysis are included (cf. Kreitz and Weihrauch [KW1], [W1]).

Each object $x \in X$ is represented by sequences ("names") $p \in \{0,1\}^{\mathbb{N}}$. Here $\mathbb{D} := \{0,1\}^{\mathbb{N}}$ with the usual product topology of the discrete topology is called the *Cantor space*. Formally a *representation* of the set X is a surjective (partial) mapping $\delta_X :\subseteq \mathbb{D} \to X$, where in general a partial mapping from X to Y is denoted by $f :\subseteq X \to Y$.

Now a function $f :\subseteq X \to Y$ is (δ_X, δ_Y)*-computable*, if there is a computable function $F :\subseteq \mathbb{D} \to \mathbb{D}$, such that $f\delta_X(p) = \delta_Y F(p)$ for all $p \in dom(f\delta_X)$, i.e. such that the following diagram commutates w.r.t. $dom(f\delta_X)$:



Furthermore computability of functions $F :\subseteq \mathbb{D} \to \mathbb{D}$ is described by Turing machines with an extended semantic (called *Type-2 machines*). A function $F :\subseteq \mathbb{D} \to \mathbb{D}$ is *computed* by a Type-2 machine M if M computes each finite prefix of the output $F(p)$ by using only a finite prefix of the input p. This implies continuity of F.

In some cases where the space X itself carries a topological structure, there are representations with nice properties. Namely for T_0-spaces X with countable base β there is a *standard representation* $\delta :\subseteq \mathbb{D} \to X$, defined by

$$\delta(p) = x :\iff U\ En(p) \text{ is a neighbourhood base of } x,$$

where $U : \mathbb{N} \to \beta$ is a numbering of the base β and

$$En : \mathbb{D} \to 2^{\mathbb{N}}, p \mapsto \{n | 10^{n+1}11 \text{ subword of } p\}$$

is the *enumeration representation* of $2^{\mathbb{N}}$.

A representation $\delta' :\subseteq I\!D \to X$ is called *admissible* if $\delta \equiv \delta'$, where "$\equiv$" is the equivalence induced by the reduction "$\leq$", defined by

$$\delta \leq \delta' :\iff (\exists G :\subseteq I\!D \to I\!D \text{ continuous})\ \delta = \delta' G.$$

A function $f :\subseteq X \to Y$ is called *relatively continuous* if there is a continuous function $F :\subseteq I\!D \to I\!D$ such that $f\delta_X = \delta_Y F$, where δ_X and δ_Y are admissible representations of X and Y.

Now the main theorem of the theory of effectivity (cf. [KW1], [W1]) states

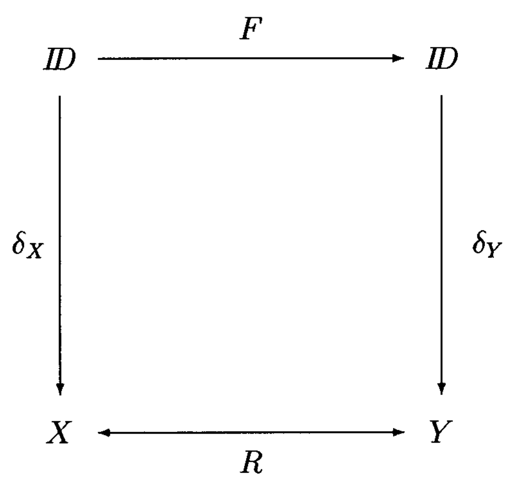
$$f \text{ continuous} \iff f \text{ relatively continuous.}$$

Especially continuity of f is a necessary condition for (δ_X, δ_Y)-computability of f.

It is a lack of this concept, as far as described, that it does not use the whole computational power of Type-2 machines. Up to here we used only extensional functions $F :\subseteq I\!D \to I\!D$ (i.e. for names $p, q \in I\!D$ with $\delta_X(p) = \delta_X(q)$ we have $\delta_Y F(p) = \delta_Y F(q)$), while Type-2 machines allow non-extensional functions F too. For non-extensional functions F the computed objects, related with F, are relations $R \subseteq X \times Y$. From another point of view these are non-deterministic operations or set-valued functions $f :\subseteq X \to 2^Y$ with

$$R = \{(x, y) | y \in f(x)\}.$$

A relation $R \subseteq X \times Y$ is called *(δ_X, δ_Y)-computable* if there is a computable function $F :\subseteq I\!D \to I\!D$ such that $(\delta_X(p), \delta_Y F(p)) \in R$ whenever $p \in dom(R\delta_X)$ (cf. Weihrauch [W1]).

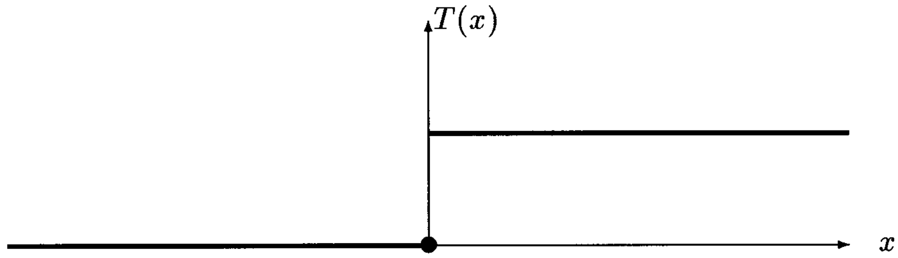


Relative continuity and (δ_X, δ_Y)-continuity are defined correspondingly.

One possible application of these relations is the approximation of discontinuous functions. For example the test

$$T : I\!R \to I\!R, x \mapsto \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1 & \text{else} \end{cases}$$

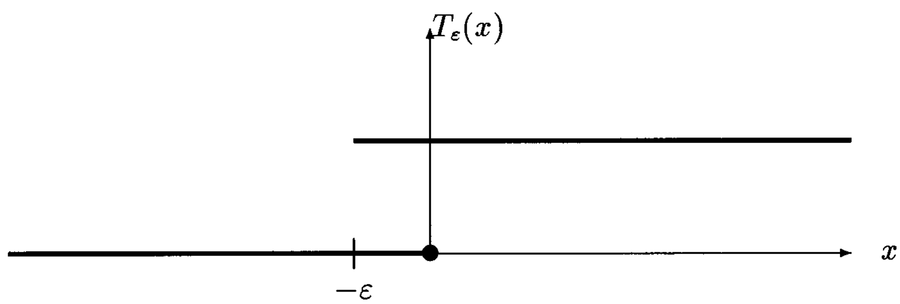
is discontinuous and hence not computable (w.r.t. admissible representations).



Nevertheless it is possible to compute the relation

$$T_\varepsilon := \{(x,0) \in I\!R \times I\!R | x \leq 0\} \cup \{(x,1) \in I\!R \times I\!R | x > -\varepsilon\}$$

for arbitrarily small $\varepsilon > 0$. (Unfortunately the complexity of the computation will increase with decreasing ε.)



Another example, which shows that non-extensional computation reaches more problems than extensional computation, is the computation of zeros of complex polynomials. The relation $ROOTS \subseteq \mathbb{C}^n \times \mathbb{C}^n$, defined by

$$ROOTS := \{(a,w) | \{w_0, ..., w_{n-1}\} \text{ is the set of zeros of } z^n + \sum_{k=0}^{n-1} a_k z^k\}$$

is continuous and computable. In fact by the results of Schönhage (cf. [Sc]) it is even polynomial time computable (cf. Ko [Ko]). On the other hand it is well known that $ROOTS$ has no continuous selector, i.e. there is no continuous function $f : \mathbb{C}^n \to \mathbb{C}^n$ with $Graph(f) \subseteq ROOTS$.

Furthermore the concept of computable relations $R \subseteq X \times Y$ is related to the notion of *operation*, widely used in constructive mathematics (cf. Bishop and Bridges [BB], Bridges and Richman [BR]).

The goal of this paper is to find a suitable notion of continuity for relations R such that the main theorem can be extended to relations. It appears that a very natural notion of continuity, used by several other authors in topology and descriptive set theory, fits our needs. Especially our continuity coincides with *lower semi-continuity* of set-valued functions. Namely a relation $R \subseteq X \times Y$ is continuous, if

$$R^{-1}(V) = \{x \in X | (\exists y \in V)(x, y) \in R\}$$

is open in $dom(R)$ for each open set $V \subseteq Y$.

Unfortunately the generalization of the main theorem to relations is not true in general.

First we restrict ourselves to relations with closed (resp. compact) images

$$R(x) = \{y \in Y | (x, y) \in R\}.$$

We prove that this assumption is not superfluous. For separable metric spaces X the first restriction is no proper restriction, because we can show that each relatively continuous relation has a continuous restriction (with the same domain) and even with compact images.

The second restriction is, that we consider only the special class of *selectable* (resp. *K-selectable*) topological spaces Y, which includes the complete separable metric ones (resp. the separable metric ones). We can even show that this restriction is necessary.

Under the two mentioned restrictions our main theorem appears as follows:

$$\begin{aligned} & R \text{ is relatively continuous} \\ \iff \quad & R \text{ has a continuous restriction (with the same domain).} \end{aligned}$$

In comparison with the main theorem for functions "being continuous" is replaced by "having a continuous restriction". This restriction appears inevitably since a relation $R \subseteq X \times Y$ which is (δ_X, δ_Y)-continuous via $F : \mathbb{D} \to \mathbb{D}$ is not necessarily represented exactly by F, i.e. in general

$$R_F := \{(\delta_X(p), \delta_Y F(p)) | p \in \mathbb{D}\}$$

is only a subrelation of R. Hence arbitrary extensions (with the same domain) of a relatively continuous relation are relatively continuous.

Additionally we investigate two notions of computability for relations, induced from the viewpoints of "relations" versus "set-valued functions".

2 Continuous Relations

It is the aim of this section to generalize the notion of continuity from functions to relations in a reasonable way. We define continuity in a point and three sorts of global continuity.

First we introduce some notations. Let X, Y be sets and $R \subseteq X \times Y$ be a relation. We define

$$R(x) := \{y \in Y \mid (x,y) \in R\},$$
$$R^{-1}(y) := \{x \in X \mid (x,y) \in R\},$$
$$R^{-1}(V) := \{x \in X \mid R(x) \cap V \neq \emptyset\},$$
$$dom(R) := \{x \in X \mid (\exists y \in Y)\ (x,y) \in R\},$$
$$range(R) := \{y \in Y \mid (\exists x \in X)\ (x,y) \in R\}.$$

In the rest of the section we assume X and Y to be topological spaces.

Definition 2.1 Let $R \subseteq X \times Y$ be a relation.

(i) Fix a point $(x,y) \in R$. R is *continuous* in (x,y)
$:\Longleftrightarrow$ ($\forall$ neighbourhoods V of y) ($\exists$ neighbourhood U of x)
$(\forall\, \hat{x} \in U \cap dom(R))\ V \cap R(\hat{x}) \neq \emptyset$.

(ii) R is *continuous*
$:\Longleftrightarrow$ R is continuous in all points $(x,y) \in R$.

(iii) R has a *continuous restriction*
$:\Longleftrightarrow$ there is a continuous relation $S \subseteq R$ with $dom(S) = dom(R)$.

(iv) R is *weakly continuous*
$:\Longleftrightarrow$ $(\forall x \in dom(R))\,(\exists y \in R(x))$ R is continuous in (x,y).

Continuity for relations as in (ii) has already been used by other authors (cf. Choquet [Ch] pp. 70 - 71 and Adamowicz [Ad] p. 82, p. 93). In later sections we will need only the continuity (ii) and the restricted continuity (iii). The continuity (ii) can be expressed in the same way as for functions.

Lemma 2.2 *A relation $R \subseteq X \times Y$ is continuous iff for any open set $V \subseteq Y$ the set $R^{-1}(V)$ is open in $dom(R)$.*

Proof. First suppose $R^{-1}(V)$ is open in $dom(R)$ for any open V. Fix an arbitrary point $(x,y) \in R$ and an open neighbourhood V of y. Since $R^{-1}(V)$ is open in $dom(R)$, R is continuous in (x,y).

Now let R be continuous. Let $V \subseteq Y$ be an open set. Fix an element $x \in R^{-1}(V)$. There is a $y \in V \cap R(x)$. Since V is a neighbourhood of y there is a neighbourhood U of x such that $(\forall \hat{x} \in U \cap dom(R)) V \cap R(\hat{x}) \neq \emptyset$. Hence the set $R^{-1}(V)$ contains a neighbourhood of x w.r.t. $dom(R)$. Thus $R^{-1}(V)$ is open in $dom(R)$. □

The definitions generalize the continuity of functions. Namely, if $f :\subseteq X \longrightarrow Y$ is a function, then obviously

$$f \text{ is continuous in } x \in dom(f) \iff graph(f) \text{ is continuous in } (x, f(x))$$

and

$$\begin{aligned} f \text{ is continuous} &\iff graph(f) \text{ is continuous} \\ &\iff graph(f) \text{ has a continuous restriction} \\ &\iff graph(f) \text{ is weakly continuous.} \end{aligned}$$

But for relations the three notions of continuity are not equivalent.

Proposition 2.3 *For relations $R \subseteq X \times Y$ the three statements*

(1) R is continuous,

(2) R has a continuous restriction,

(3) R is weakly continuous

fulfill the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), *but in general* (1) $\not\Leftarrow$ (2) *and* (2) $\not\Leftarrow$ (3).

Proof. "(1) $\Rightarrow$ (2)": trivial.

"(2) $\Rightarrow$ (3)": trivial.

"(1) $\not\Leftarrow$ (2)": The relation $R := \{(x,0) \mid x \leq 1\} \cup \{(x,1) \mid x > 0\} \subseteq \mathbb{R} \times \mathbb{R}$ is not continuous, but its restriction $R \setminus \{(1,0)\}$ is continuous.

"(2) $\not\Leftarrow$ (3)": The relation $R \subseteq \mathbb{R} \times \mathbb{R}$, defined by

$$\begin{aligned} R := \; & \{(x,0) \mid x \leq 0\} \cup \{(x,-1) \mid x > 0\} \\ & \cup \{(x,0) \mid x > 0 \text{ and } x \in \mathbb{Q}\} \cup \{(x,x) \mid x > 0 \text{ and } x \in \mathbb{R} \setminus \mathbb{Q}\} \end{aligned}$$

is weakly continuous, but it does not have a continuous restriction. □

Finally we note that the concatenation of two continuous relations does not need to be continuous.

Definition 2.4 Let X, Y, Z be sets and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ be relations.

$$S \circ R := \{(x,z) \in X \times Z \mid (\exists y)\, (x,y) \in R \text{ and } (y,z) \in S\}.$$

Example 2.5 The relations

$$R := \{(x,x) \mid 0 \leq x < 1\} \cup \{(x,2) \mid 0 < x < 1\} \subseteq I\!R \times I\!R \quad \text{and}$$

$$S := \{(0,0),(2,1)\} \subseteq I\!R \times I\!R$$

are continuous while the relation

$$S \circ R = \{(0,0)\} \cup \{(x,1) \mid 0 < x < 1\}$$

is not even weakly continuous.

But if the first relation is a function then the concatenation is continuous.

Lemma 2.6 *Let X, Y, Z be topological spaces, $f :\subseteq X \longrightarrow Y$ be a continuous function and $R \subseteq Y \times Z$ be a continuous relation. Then the relation $R \circ f := R \circ Graph(f)$ is continuous.*

The proof is left to the reader. In fact the characterization of continuous relations as lower semi-continuous set-valued functions in the next section immediately implies the lemma.

3 Set-valued Functions

In this section we will grasp relations as set-valued functions. This equivalent point of view has some advantages, especially for relations with closed images.

For a set X we define

$$\mathcal{P}(X) := \{A \subseteq X | A \neq \emptyset\}$$

and if furthermore X is a topological space we define

$$\mathcal{A}(X) := \{A \subseteq X | A \neq \emptyset \text{ and } A \text{ closed}\}$$

and the set of all compact subsets

$$\mathcal{K}(X) := \{A \subseteq X | A \neq \emptyset \text{ and } A \text{ compact}\}.$$

Here a topological space is called *compact*, if it is a Hausdorff space and each open cover contains a finite subcover.

With any relation $R \subseteq X \times Y$ we associate a set-valued function

$$\hat{R} :\subseteq X \to \mathcal{P}(Y), x \mapsto R(x)$$

with $dom(\hat{R}) = dom(R)$. Obviously this association gives a bijection between the set of relations $R \subseteq X \times Y$ and the set of functions $F :\subseteq X \to \mathcal{P}(Y)$. Now we introduce a topology for the set $\mathcal{P}(X)$ of a topological space X.

Definition 3.1 Let X be a topological space. The *lower topology* $\tau_<$ for the set $\mathcal{P}(X)$ is induced by the subbase

$$\{B_<(U) | U \subseteq X \text{ open}\}$$

with

$$B_<(U) := \{A \subseteq X | A \cap U \neq \emptyset\}.$$

This topology is called *λ-topology* (cf. [Ku] Ch. I §18 II p. 175) or *lower semi-finite topology* (cf. [Mi] Def. 9.1 p. 179).

The first observation is that our notion of continuity for relations coincides with the continuity of the corresponding set-valued function w.r.t. $\tau_<$. A function $F :\subseteq X \to \mathcal{P}(Y)$ is continuous w.r.t. $\tau_<$ iff for any open set $V \subseteq Y$ the set

$$\{x \in X | F(x) \cap V \neq \emptyset\}$$

is open in $dom(F)$. Such functions F are called *lower semi-continuous* (cf. [Ku] Ch. I §18 I p. 173, [En] 1.7.17 p. 63).

Proposition 3.2 *A relation $R \subseteq X \times Y$ is continuous iff the corresponding function $\hat{R} :\subseteq X \longrightarrow \mathcal{P}(Y)$ is lower semi-continuous.*

Proof. Since
$$R^{-1}(V) = \{x \in X | R(x) \cap V \neq \emptyset\}$$
for all $V \subseteq Y$, all we need follows from Lemma 2.2. □

The connection between a relation and it's associated function can be expressed in another way with the help of a choice relation.

Definition 3.3 Let X be a topological space. Then the *choice relation* of X is defined by
$$Choice := \{(A, x) \in \mathcal{P}(X) \times X | x \in A\}.$$

Obviously $\hat{Choice}$ is the identity of $\mathcal{P}(X)$ and therefore $\hat{Choice}$ is lower semi-continuous and $Choice$ is continuous. For each relation $R \subseteq X \times Y$ we have
$$Choice \circ \hat{R} = R.$$

Now we want to concentrate on relations with closed images. Let X be a set and Y a topological space. We will say that a relation $R \subseteq X \times Y$ has *closed images* if $R(x)$ is closed in Y for all $x \in dom(R)$. For such relations we have $range(\hat{R}) \subseteq \mathcal{A}(Y)$. With each relation $R \subseteq X \times Y$ we can associate it's *image closure*
$$\overline{R} := \{(x, y) \in X \times Y | y \in \overline{R(x)}\},$$
where $\overline{R(x)}$ means the topological closure of $R(x)$ in Y. In the situation where the notion of restricted continuity is used in connection with relations with closed images, it is useful to have the following proposition.

Proposition 3.4 *Let X, Y be topological spaces and let $R \subseteq X \times Y$ be a relation with closed images which has a continuous restriction. Then R especially has a continuous restriction with closed images.*

It is easy to construct such a restriction: if S is a continuous restriction of R then $\overline{S}$ is a restriction of R with closed images. By the next lemma $\overline{S}$ is continuous too.

Lemma 3.5 *Let X, Y be topological spaces and let $R \subseteq X \times Y$ be a relation. Then we have:*
$$R \text{ continuous} \Longrightarrow \overline{R} \text{ continuous}.$$

Proof. Fix $(x, y) \in \overline{R}$. We have to show that $\overline{R}$ is continuous in (x, y). If $(x, y) \in R$ then the assertion follows immediately from continuity of R. If $(x, y) \in \overline{R} \setminus R$ then $y \in \partial R(x) \setminus R(x)$. Fix an open neighbourhood V of y. Then there is a $y' \in R(x) \cap V$ and since R is continuous in (x, y') there exists a neighbourhood U of x with $R(\hat{x}) \cap V \neq \emptyset$ for all $\hat{x} \in U$. Hence $\overline{R}(\hat{x}) \cap V \neq \emptyset$ for all $\hat{x} \in U$ and $\overline{R}$ is continuous in (x, y). □

We will close this section with a proposition which states some nice properties of the space $(\mathcal{P}(X), \tau_<)$, resp. $(\mathcal{A}(X), \tau_<)$ (understood as a subspace of $(\mathcal{P}(X), \tau_<)$).

Proposition 3.6 *Let X be a topological space with base β. Then we have:*

(1) $(\mathcal{A}(X), \tau_<)$ is a T_0-space,

(2) $\{B_<(U) | U \in \beta\}$ is a subbase of $(\mathcal{P}(X), \tau_<)$.

Proof.

(1) Let $A, B \in \mathcal{A}(X)$ such that $A \neq B$. W.l.o.g. $A \setminus B \neq \emptyset$. Hence

$$A \in B_<(B^c) \text{ and } B \notin B_<(B^c).$$

(2) For each open $U \subseteq X$ there is a $\mathcal{U} \subseteq \beta$ with $U = \cup\mathcal{U}$, hence

$$B_<(U) = \bigcup_{V \in \mathcal{U}} B_<(V).$$

□

If X is a Hausdorff space then $\mathcal{K}(X) \subseteq \mathcal{A}(X)$ and by (1) the space $(\mathcal{K}(X), \tau_<)$ is a T_0-space. If X has a countable base then by (2) the spaces $(\mathcal{P}(X), \tau_<)$, $(\mathcal{A}(X), \tau_<)$ and $(\mathcal{K}(X), \tau_<)$ have countable bases too.

4 Relative Continuity

We use the following definition in order to define relative continuity for relations between T_0-spaces with countable bases.

Definition 4.1 Let X_i be topological spaces, M_i be sets and $\delta_i :\subseteq X_i \longrightarrow M_i$ be maps ($i = 1, 2$). A relation $R \subseteq M_1 \times M_2$ is (δ_1, δ_2)-*continuous* iff there is a continuous function $F :\subseteq X_1 \longrightarrow X_2$ such that

$$(\forall p \in dom(R\delta_1))\ \delta_2 F(p) \in R\delta_1(p).$$

Definition 4.2 Let (M_i, τ_i) $(i = 1, 2)$ be T_0-spaces with countable bases. A relation $R \subseteq M_1 \times M_2$ is called *relatively continuous* iff there are τ_i-admissible representations $\delta_i :\subseteq I\!D \longrightarrow M_i$ such that R is (δ_1, δ_2)-continuous.

By the next proposition the property of a relation to be relatively continuous can be checked by using arbitrary τ_i-admissible representations.

Proposition 4.3 *Let (M_i, τ_i) $(i = 1, 2)$ be T_0-spaces with countable bases. A relation $R \subseteq M_1 \times M_2$ is relatively continuous iff R is (δ_1, δ_2)-continuous for all τ_i-admissible representations $\delta_i :\subseteq I\!D \longrightarrow M_i$.*

The proposition is an immediate consequence of the next lemma.

Lemma 4.4 *Let X_i, X'_i be topological spaces $(i = 1, 2)$, M_i be sets and $\delta_i :\subseteq X_i \longrightarrow M_i, \delta'_i :\subseteq X'_i \longrightarrow M_i$ be maps satisfying $\delta'_1 \leq \delta_1$, $\delta_2 \leq \delta'_2$. Then for a relation $R \subseteq M_1 \times M_2$ we have*

$$R \text{ is } (\delta_1, \delta_2) - \text{continuous} \ \Rightarrow\ R \text{ is } (\delta'_1, \delta'_2) - \text{continuous}.$$

The proof of the lemma is left to the reader.

It is the aim of the paper to investigate the connection between relative continuity and continuity. By the following proposition a relatively continuous relation has a continuous restriction. The inverse implication is true in special cases and will be considered in the next sections.

Proposition 4.5 *Let X and Y be T_0-spaces with countable bases. If $R \subseteq X \times Y$ is a relatively continuous relation, then R has a continuous restriction.*

Proof. Fix an admissible open representation $\delta_X :\subseteq I\!D \longrightarrow X$ — e.g. the standard representation — and an admissible representation $\delta_Y :\subseteq I\!D \longrightarrow Y$. Let $F :\subseteq I\!D \longrightarrow I\!D$ be a function such that $(\forall p \in dom(R\delta_X))\ \delta_Y F(p) \in R\delta_X(p)$. By applying the next lemma to the function $F' := F|_{dom(R\delta_X)}$ we see that the relation

$$R_{F'} := \{(\delta_X(p), \delta_Y F'(p)) \mid p \in dom(\delta_Y F')\}$$

is continuous. Since obviously $dom(R_{F'}) = dom(R)$ and $R_{F'} \subseteq R$ the assertion is proved. □

Lemma 4.6 *Let X, Y be topological spaces, $\delta_X :\subseteq I\!D \longrightarrow X$ be an open representation and $\delta_Y :\subseteq I\!D \longrightarrow Y$ be a continuous representation. Let $R \subseteq X \times Y$ be a relation. If R is (δ_X, δ_Y)-continuous via $F :\subseteq I\!D \to I\!D$ with $dom(F) = dom(R\delta_X)$ then*

$$R_F := \{(\delta_X(p), \delta_Y F(p)) \mid p \in dom(\delta_Y F)\}$$

is a continuous restriction of R.

Proof. Fix a point $(x, y) = (\delta_X(v), \delta_Y F(v)) \in R_F$. If V is a neighbourhood of y then there is an open set $U' \subseteq I\!D$ such that

$$(\delta_Y F)^{-1}(V) = U' \cap dom(\delta_Y F).$$

Since $dom(\delta_Y F) = dom(F) = \delta_X^{-1}(dom(R)) = \delta_X^{-1}\delta_X(dom(\delta_Y F))$ the set

$$\begin{aligned} U &:= \delta_X(\delta_Y F)^{-1}(V) \\ &= \delta_X(U' \cap \delta_X^{-1}\delta_X(dom(\delta_Y F))) \\ &= \delta_X(U') \cap \delta_X(dom(\delta_Y F)) \\ &= \delta_X(U') \cap dom(R_F) \end{aligned}$$

is open in $dom(R_F)$ and $x \in U$. Furthermore for all $\hat{x} \in U$ one has $V \cap R_F(\hat{x}) \neq \emptyset$. □

We close the section with two remarks on the concatenation of relations. The Example 2.5 also shows that the concatenation of two relatively continuous relations does not need to be relatively continuous. But in analogy to Lemma 2.6 one can easily prove:

Lemma 4.7 *Let X, Y, Z be T_0-spaces with countable bases, $f :\subseteq X \longrightarrow Y$ be a relatively continuous function and $R \subseteq Y \times Z$ be a relatively continuous relation. Then the relation $R \circ f$ is relatively continuous.*

5 Selectable Spaces and the Main Theorem

In this section we will show that the main theorem of the theory of effectivity can be extended from functions to relations with closed images. Indeed this is true only for a special class of topological spaces. We will call them *selectable* in this paper.

In this section a *zero-dimensional topological space* X is understood to be a T_3-space with a countable base of clopen (i.e. closed and open) sets (cf. [Ku]).

First a basic notion is introduced.

Definition 5.1 Let $R \subseteq X \times Y$ be a relation. A *selector* of R is a function $s :\subseteq X \to Y$ with $dom(s) = dom(R)$ and $graph(s) \subseteq R$.

This means $s(x) \in R(x)$ for all $x \in dom(s)$ if s is a selector of R. Now we are ready to define selectable spaces.

Definition 5.2 A topological space Y is called *selectable* if for each zero-dimensional space X and for each continuous relation $R \subseteq X \times Y$ with closed images there is a continuous selector s of R.

First we mention that it suffices to require the existence of a selector for a "universal" zero-dimensional space. As a universal zero-dimensional space we use the Cantor space $\mathbb{D} := \{0,1\}^{\mathbb{N}}$ with the usual product topology of the discrete space $\{0,1\}$. $\mathbb{D}$ is homeomorphic to Cantor's discontinuum. Since there exists a homeomorphism $h :\subseteq \mathbb{D} \to X$ for each zero-dimensional space (cf. [Ku] Ch. II §26 IV Th. 2 p. 285) the proof of the following lemma is obvious.

Lemma 5.3 *A topological space Y is selectable iff for each continuous relation $R \subseteq \mathbb{D} \times Y$ with closed images there is a continuous selector s of R.*

Now we can formulate a characterisation of the selectable T_0-spaces with countable bases which shows that they are exactly those spaces for which the main theorem becomes true. The main idea of the proof is to factorize each continuous relation $S \subseteq X \times Y$ with closed images into it's functional part $\hat{S} :\subseteq X \to \mathcal{A}(Y)$ and the choice relation $Choice \subseteq \mathcal{A}(Y) \times Y$. We assume that $\mathcal{A}(Y)$ is equipped with the topology $\tau_<$ as far as not declared differently. Proposition 3.6 shows that $(\mathcal{A}(Y), \tau_<)$ is a T_0-space with countable base if Y is. Hence there is a $\tau_<$-admissible representation $\delta^<_{\mathcal{A}(Y)}$ of $\mathcal{A}(Y)$.

Theorem 5.4 *Let Y be a T_0-space with countable base. Then the following statements are equivalent:*

(1) *Y is selectable.*

(2) *Choice* $\subseteq \mathcal{A}(Y) \times Y$ *is relatively continuous.*

(3) *For all* T_0*-spaces* X *with countable base and all relations* $R \subseteq X \times Y$ *with closed images the following is equivalent:*

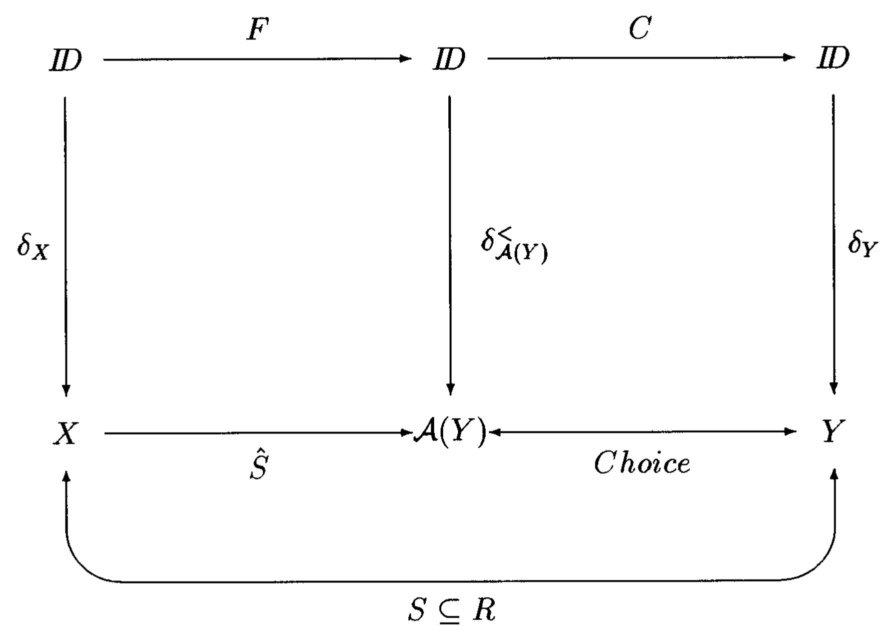
  (a) R *is relatively continuous,*

  (b) R *has a continuous restriction.*

Proof. Let $\delta_Y, \delta^<_{\mathcal{A}(Y)}$ be fixed admissible representations of Y resp. $\mathcal{A}(Y)$.

"(1) $\Longrightarrow$ (2)" Let Y be selectable. Since $\delta^<_{\mathcal{A}(Y)}$ is continuous the relation $Choice \circ \delta^<_{\mathcal{A}(Y)} \subseteq \mathbb{D} \times Y$ is continuous by Lemma 2.6. Since it obviously has closed images there is a continuous selector $c :\subseteq \mathbb{D} \to Y$ of $Choice \circ \delta^<_{\mathcal{A}(Y)} \subseteq \mathbb{D} \times Y$. By the main theorem for functions there is a continuous function $C :\subseteq \mathbb{D} \to \mathbb{D}$ such that c is (id, δ_Y)-continuous via C and hence $Choice$ is $(\delta^<_{\mathcal{A}(Y)}, \delta_Y)$-continuous via C.

The following diagram explains the situation:



"(2) $\Longrightarrow$ (3)" Let $Choice$ be relatively continuous. Then $Choice$ is $(\delta^<_{\mathcal{A}(Y)}, \delta_Y)$-continuous via a continuous function $C :\subseteq \mathbb{D} \to \mathbb{D}$ by Proposition 4.3. Let X be a topological space with admissible representation δ_X and let $R \subseteq X \times Y$ be a relation with closed images. "$(a) \Longrightarrow (b)$" This implication holds in general by Proposition 4.5. "$(b) \Longrightarrow (a)$" Let S be a continuous restriction of R. By Proposition 3.4 S can be assumed to have closed images. By Proposition

3.2 $\hat{S} :\subseteq X \to \mathcal{A}(Y)$ is lower semi-continuous and $Choice \circ \hat{S} = S$. Since $\delta^<_{\mathcal{A}(Y)}$ is admissible the main theorem for functions yields a continuous function $F :\subseteq I\!D \to I\!D$ such that $\hat{S}$ is $(\delta_X, \delta^<_{\mathcal{A}(Y)})$-continuous via F. Hence S and therefore R is (δ_X, δ_Y)-continuous via CF.

"(3) $\Longrightarrow$ (1)" Let $R \subseteq I\!D \times Y$ be a continuous relation with closed images. By the previous lemma it suffices to show that R has a continuous selector s. By assumption (3) R is (id, δ_Y)-continuous via a continuous function $G :\subseteq I\!D \to I\!D$. W.l.o.g. we can assume $dom(G) = dom(R)$. Hence $s := \delta_Y G$ is a continuous selector of R. □

This characterization especially shows that selectable T_0-spaces are exactly those spaces fulfilling a kind of weak effective axiom of choice (2). Michael (cf. [Mi]) has investigated spaces fulfilling a strong axiom of choice. He defined *S_1-spaces* to be topological spaces such that $Choice \subseteq \mathcal{A}(Y) \times Y$ itself has a continuous selector which is more than being relatively continuous.

Up to now it is not clear that there are any selectable spaces. In the next section we will prove that complete separable metric spaces are selectable (cf. Corollary 6.4). This result can also be seen as a special case of the Selection theorem of Kuratowski and Ryll-Nardzewski (cf. [KR] and [Ku] Ch. IV §43 IX) applied to the field L of clopen subsets of Cantor's space. In this case the set L_σ of all countable unions of elements of L is the topology of Cantor's space. Now we are ready to formulate the main theorem:

Corollary 5.5 (The main theorem) *Let X be a T_0-space with a countable base and let Y be a complete separable metric space. If $R \subseteq X \times Y$ is a relation with closed images then the following is equivalent:*

(1) R is relatively continuous,

(2) R has a continuous restriction.

As a corollary we gain a connection between the two kinds of continuity used in the proof of the main theorem, namely (δ_X, δ_Y)-continuity of relations $R \subseteq X \times Y$ and $(\delta_X, \delta^<_{\mathcal{A}(Y)})$-continuity of $\hat{R}$.

Corollary 5.6 *Let X be a T_0-space with a countable base and let Y be a complete separable metric space. If $R \subseteq X \times Y$ is a relation with closed images then the following holds:*

$$\hat{R} \text{ is relatively continuous} \Longrightarrow R \text{ is relatively continuous.}$$

The converse is not true in general as Example 6.6 in the next section shows.

By Corollary 6.4 all complete separable metric spaces are selectable. The completeness assumption is important since by the next example even the separable metric space $\mathbb{Q}$ is not selectable.

Example 5.7 *$\mathbb{Q}$ (with the subspace topology induced by $\mathbb{R}$) is not selectable.*

Proof. By Theorem 5.4 we have to show that the relation $Choice \subseteq \mathcal{A}(\mathbb{Q})\times\mathbb{Q}$ is not relatively continuous. Let us assume that $Choice$ is relatively continuous, i.e. let us assume that there is continuous function $F :\subseteq \mathbb{D} \longrightarrow \mathbb{D}$ with $\delta_{\mathbb{Q}} F(p) \in Choice \circ \delta^{<}_{\mathcal{A}(\mathbb{Q})}(p)$ for all $p \in dom(Choice \circ \delta^{<}_{\mathcal{A}(\mathbb{Q})})$ where $\delta_{\mathbb{Q}}$ and $\delta^{<}_{\mathcal{A}(\mathbb{Q})}$ are standard representations of $\mathbb{Q}$ and of $(\mathcal{A}(\mathbb{Q}), \tau_{<})$ w.r.t. a numbering U of a base of $\mathbb{Q}$. We shall construct a name $p \in {\delta^{<}_{\mathcal{A}(\mathbb{Q})}}^{-1}(\mathbb{Q})$ such that for arbitrary $n \geq 1$ there is a finite prefix $w_n \sqsubseteq p$ with $\delta_{\mathbb{Q}} F(w_n\mathbb{D}) \subseteq \mathbb{Q} \setminus \mathbb{Q}^{(n)}$ where

$$\mathbb{Q}^{(n)} := \{\frac{k}{l} \mid k \in \mathbb{Z},\ l \in \{1, \ldots, n\}\} \quad \text{for } n \geq 1.$$

This implies $\delta_{\mathbb{Q}} F(p) \notin \mathbb{Q}$, contradicting the assumption.

Fix an arbitrary name $q \in {\delta^{<}_{\mathcal{A}(\mathbb{Q})}}^{-1}(\mathbb{Q})$. Let $q^{(i)}$ denote the i-th subword of q of the form $10^{m+1}11$ ($m \in \mathbb{N}$) beginning with $i = 0$. Since $\mathbb{Q} = \delta^{<}_{\mathcal{A}(\mathbb{Q})}(q)$ the name q must contain infinitely many such subwords. We construct p as the limit of a sequence $w_0 \sqsubseteq w_1 \sqsubseteq w_2 \sqsubseteq \ldots$ which is defined inductively by inserting finite words between the $q^{(i)}$.

Set $w_0 := q^{(0)}$.

For the inductive step we assume w_n to be given and proceed as follows. For any open set $V \in UEn(w_n 0^\omega)$ we choose a rational number $r(V) \in V \cap \mathbb{Q} \setminus \mathbb{Q}^{(n+1)}$. The finite set $A_n := \{r(V) \mid V \in UEn(w_n 0^\omega)\}$ is certainly closed. Furthermore there is a name $p_n \in w_n\mathbb{D} \cap {\delta^{<}_{\mathcal{A}(\mathbb{Q})}}^{-1}(A_n)$ in $w_n\mathbb{D}$ for A_n. Since $\delta_{\mathbb{Q}} F(p_n) \in A_n \subseteq \mathbb{Q} \setminus \mathbb{Q}^{(n+1)}$ and $\delta_{\mathbb{Q}} F$ is continuous there is prefix $v_n \sqsubseteq p_n$ with $\delta_{\mathbb{Q}} F(v_n\mathbb{D}) \subseteq \mathbb{Q} \setminus \mathbb{Q}^{(n+1)}$. We can assume w.l.o.g. that $w_n \sqsubseteq v_n$. Now set $w_{n+1} := v_n q^{(n)}$.

Thus we have constructed a sequence $w_n, n \in \mathbb{N}$, with

(1) $(\forall n \in \mathbb{N})\ w_n \sqsubseteq w_{n+1}$,

(2) $(\forall n \geq 1)\ \delta_{\mathbb{Q}} F(w_n\mathbb{D}) \subseteq \mathbb{Q} \setminus \mathbb{Q}^{(n)}$,

(3) $(\forall n \in \mathbb{N})\ q^{(n)}$ is a subword of w_n.

By (1) the limit $p := \lim_{n\to\infty} w_n$ exists, and by (3) $p \in {\delta^{<}_{\mathcal{A}(\mathbb{Q})}}^{-1}(\mathbb{Q})$. But (2) implies $\delta_{\mathbb{Q}} F(p) \notin \mathbb{Q}$. Contradiction!

Hence $Choice \subseteq \mathcal{A}(\mathbb{Q}) \times \mathbb{Q}$ cannot be relatively continuous. □

We finish this section with some remarks on relations with compact images. If one considers only relations with compact images instead of relations with closed images one obtains very similar results. We sketch them. The reader can easily verify them by going through the proofs in this section.

In analogy to Definition 5.2 we call a topological space Y *K-selectable* if for each zero-dimensional space X and for each continuous relation $R \subseteq X \times Y$ with compact images there is a continuous selector of R. Lemma 5.3 and Theorem 5.4 hold true if one replaces "selectable" by "K-selectable", "closed images" by "compact images" and considers the restricted choice-relation $Choice \subseteq \mathcal{K}(Y) \times Y$. Additionally we have to assume that Y is a T_2-space in Theorem 5.4, otherwise $\mathcal{K}(Y)$ is not necessarily a subset of $\mathcal{A}(Y)$. Corollaries 5.5 and 5.6 remain true if one replaces "complete separable" by "separable", "closed images" by "compact images" and $\mathcal{A}(Y)$ by $\mathcal{K}(Y)$.

6 Computable Relations

In this section we will investigate computable relations. First we introduce computability w.r.t. representations.

Definition 6.1 Let M_i be sets with representations $\delta_i :\subseteq I\!D \to M_i$, $(i = 1, 2)$. A relation $R \subseteq M_1 \times M_2$ is (δ_1, δ_2)-*computable* iff there is a computable function $F :\subseteq I\!D \to I\!D$ such that $(\forall p \in dom(R\delta_1))\ \delta_2 F(p) \in R\delta_1(p)$.

From the definition and Proposition 4.5 we gain the fact, that continuity is a necessary condition for computability.

Corollary 6.2 *Let X and Y be T_0-spaces with countable bases. Let $\delta_X :\subseteq I\!D \to X$ and $\delta_Y :\subseteq I\!D \to Y$ be admissible representations. For each relation $R \subseteq X \times Y$ the following holds:*

$$R \textit{ is } (\delta_X, \delta_Y)\textit{-computable} \Longrightarrow R \textit{ has a continuous restriction.}$$

The next theorem implies that complete separable metric spaces are selectable. We will prove a stronger result by showing that a computable axiom of choice is true for this class. Therefore we have to fix the admissible representations and the corresponding base numberings.

For a separable metric space (Y, d) with a numbering $\alpha : I\!N \to D$ of a dense subset $D \subseteq Y$ we choose

$$U : I\!N \to \beta, < n, k > \mapsto B(\alpha(n), 2^{-k})$$

as a base numbering. Here $B(x, \varepsilon) := \{y \in Y | d(x, y) < \varepsilon\}$ denotes the open ball with center x and radius ε in Y. Now by

$$\delta_Y :\subseteq I\!D \to Y, \delta_Y(p) = x :\Longleftrightarrow U(En(p)) \text{ is a neighbourhood base of } x$$

an admissible standard representation δ_Y of Y is defined and by

$$\delta^{<}_{\mathcal{A}(Y)} :\subseteq I\!D \to \mathcal{A}(Y), \delta^{<}_{\mathcal{A}(Y)}(p) = A :\Longleftrightarrow En(p) = \{n | A \in B_<(U_n)\}$$

an admissible standard representation of $(\mathcal{A}(Y), \tau_<)$ is defined, where En is the enumeration representation of $2^{I\!N}$. (For the admissibility of $\delta^{<}_{\mathcal{A}(Y)}$ cf. Weihrauch [W3]. For the special case $Y = I\!R$ an equivalent representation was introduced in Kreitz and Weihrauch [KW2].) For this section the standard representations are assumed to be fixed for each separable metric space Y, as far as not declared differently. We will call (Y, d, D, α) a *semi-computable metric space* if furthermore

$$\{(n, k, q) \in I\!N \times I\!N \times \mathbb{Q} | d(\alpha(n), \alpha(k)) < q\}$$

is recursively enumerable (cf. Weihrauch [W2]).

Proposition 6.3 *Let (Y, d, D, α) be a semi-computable complete separable metric space. Then $Choice \subseteq \mathcal{A}(Y) \times Y$ is $(\delta^{<}_{\mathcal{A}(Y)}, \delta_Y)$-computable.*

Proof. Let $p \in dom(\delta^{<}_{\mathcal{A}(Y)})$ with $A := \delta^{<}_{\mathcal{A}(Y)}(p)$. We can interpret p as a sequence $(< n_i, k_i >)_{i \in \mathbb{N}}$ of numbers of neighbourhoods. We have to compute a sequence $(< n'_i, k'_i >)_{i \in \mathbb{N}}$ such that the corresponding name $q \in \mathbb{D}$ fulfills $x := \delta_Y(q) \in A$. First we compute a sequence $(m_i)_{i \in \mathbb{N}}$ of numbers recursively:

(1) Set $m_0 := \min\{j | k_j > 1\}$.

(2) Choose j and set $m_{i+1} := j$ such that

   (a) $k_j > k_{m_i}$,

   (b) $d(\alpha(n_j), \alpha(n_{m_i})) < 2^{-k_{m_i}} - 2^{-k_j}$.

It is possible to find a j such that (b) holds since (Y, d, D, α) is a semi-computable metric space and for each open set $U \subseteq Y$ with $A \in B_<(U)$, i.e. $A \cap U \neq \emptyset$ there are arbitrarily small neighbourhoods $U_n \subseteq U$ with $A \cap U_n \neq \emptyset$, i.e. $A \in B_<(U_n)$.
Conditions (a) and (b) imply

$$B(\alpha(n_{m_{i+1}}), 2^{-k_{m_{i+1}}}) \subset B(\alpha(n_{m_i}), 2^{-k_{m_i}}).$$

Hence by

$$A_i := \overline{B(\alpha(n_{m_i}), 2^{-k_{m_i}})} \cap A$$

a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ of non-empty closed sets with $diam(A_i) < 2^{-i}$ is defined and Cantor's theorem yields a $x \in A$ with $\{x\} = \bigcap_{i=0}^{\infty} A_i$, since Y is complete. Finally by

$$V_i := B(\alpha(n_{m_i}), 2^{-k_{m_i}+1})$$

a sequence of open sets is defined which forms a neighbourhood base of x and by

$$< n'_i, k'_i > := < n_{m_i}, k_{m_i} - 1 >$$

a name $q \in \mathbb{D}$ with $\delta_Y(q) = x$ is induced. Obviously the corresponding operator $F :\subseteq \mathbb{D} \to \mathbb{D}$, mapping p to q as described by the algorithm, is computable and $Choice$ is $(\delta^{<}_{\mathcal{A}(Y)}, \delta_Y)$-computable via F. □

The same proof shows:

Corollary 6.4 *Complete separable metric spaces are selectable.*

From the previous proposition and from $Choice \circ \hat{R} = R$ we also gain a connection between the computability properties of R:

Corollary 6.5 *Let X be a T_0-space with a countable base and an admissible representation δ_X, let (Y, d, D, α) be a semi-computable complete separable metric space. If $R \subseteq X \times Y$ is a relation with closed images then the following holds:*

$$\hat{R} \textit{ is } (\delta_X, \delta^{<}_{\mathcal{A}(Y)})\textit{-computable} \Longrightarrow R \textit{ is } (\delta_X, \delta_Y)\textit{-computable.}$$

An easy example shows that the converse is not true in general.

Example 6.6 Let $(I\!R, d, \mathbb{Q}, \alpha)$ be the standard semi-computable metric space and let

$$R := I\!R \times \{0\} \cup \mathbb{Q} \times [0;1] \text{ and } R' := I\!R \times \{0\}.$$

Obviously

(1) R is not continuous, but R has the continuous restriction R',

(2) R is $(\delta_{\mathbf{R}}, \delta_{\mathbf{R}})$-computable since R' is,

(3) $\hat{R}$ is not continuous and hence not $(\delta_{\mathbf{R}}, \delta^{<}_{\mathcal{A}(\mathbf{R})})$-continuous and -computable.

□

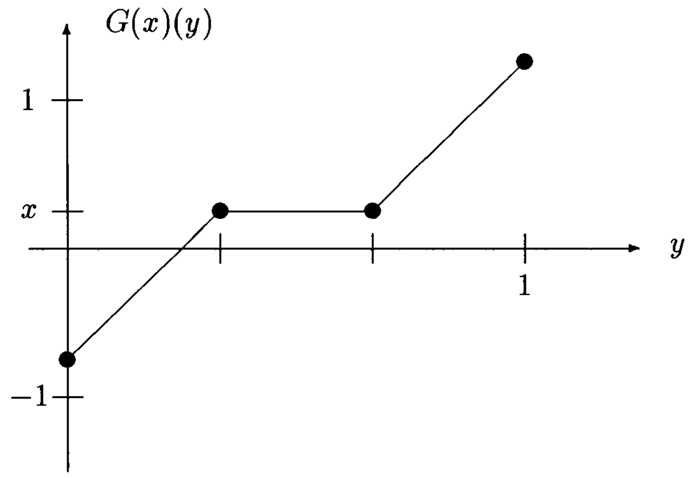
Again we make some remarks on relations with compact images. The completeness assumption in Proposition 6.3 is not necessary in the case of compact images, i.e. if (Y, d, D, α) is a semi-computable separable metric space then $Choice \subseteq \mathcal{K}(Y) \times Y$ is $(\delta^{<}_{\mathcal{K}(Y)}, \delta_Y)$-computable. The same is true for Corollaries 6.4 and 6.5, where in Corollary 6.5 one has to use $\delta^{<}_{\mathcal{K}(Y)}$ instead of $\delta^{<}_{\mathcal{A}(Y)}$. Finally the relation R in Example 6.6 already has compact images.

The next example shows how the necessary topological condition 6.2 for computability can be used to prove that certain relations are non-computable (cf. Weihrauch [W1] Th. 3.8.18 p. 495).

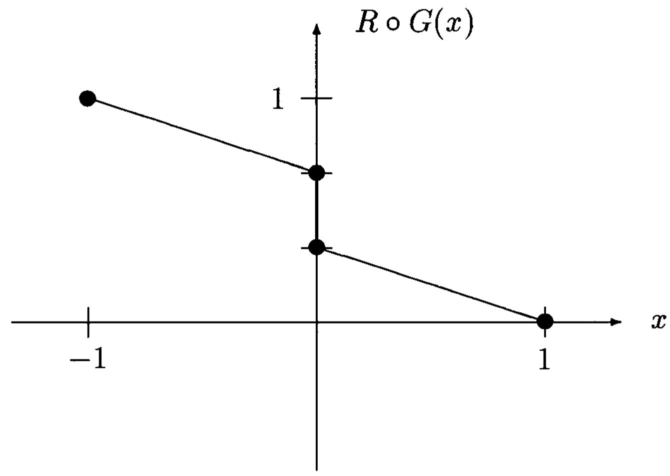
Example 6.7 Let $R \subseteq \mathcal{C}[0,1] \times I\!R$ be defined by

$$R := \bigcup_{f \in \mathcal{C}[0,1]} \{f\} \times f^{-1}(0).$$

Let $G : I\!R \to \mathcal{C}[0,1]$ be the mapping such that $G(x)$ is the polygon with the vertices $(0, x-1)$, $(1/3, x)$, $(2/3, x)$, $(1, x+1)$. Obviously G is continuous. The next figure shows $G(x)$ for $x = 1/4$.



Then $R \circ G \subseteq I\!R \times I\!R$, as illustrated in the next figure, obviously has no continuous restriction with domain $[-1, 1]$.



Hence by Lemma 2.6 R has no total continuous restriction too and hence R is not computable w.r.t. admissible representations. □

Especially $\hat{R}$ is not $(\delta_{C[0,1]}, \delta^{<}_{\mathcal{A}(I\!R)})$-computable by Corollary 6.5. Nevertheless $\hat{R}$ is $(\delta_{C[0,1]}, \delta^{>}_{\mathcal{A}(I\!R)})$-computable, where $\delta^{>}_{\mathcal{A}(I\!R)} :\subseteq I\!D \to \mathcal{A}(I\!R)$ is a $\tau_{>}$-admissible representation and $\tau_{>}$ is the *upper topology* induced by the subbase

$$\{B_{>}(U) | U \subseteq I\!R \text{ open}\} \text{ with } B_{>}(U) := \{A \in \mathcal{A}(I\!R) | A \subseteq U\}.$$

With these notations we define

$$\delta^{>}_{\mathcal{A}(I\!R)}(p) = A : \Longleftrightarrow En(p) = \{n | A \in B_{>}(U_n)\}.$$

The proof of the $(\delta_{C[0,1]}, \delta^{>}_{\mathcal{A}(I\!R)})$-computability of R is omitted.

7 Continuous Relations with Non-closed Images

First we give two examples which show that continuous relations with non-closed images do not need to be relatively continuous, even if the spaces are complete separable metric spaces.

Example 7.1 *The relation*

$$R := (\mathbb{D} \setminus W) \times W \cup W \times (\mathbb{D} \setminus W) \qquad \text{where } W := \{0,1\}^* 0^\omega$$

is continuous, but it does not have a continuous selector and it is not relatively continuous.

Proof. The relation R is obviously continuous.

The set $\mathbb{D} \setminus W$ is a G_δ-set. If there were a continuous selector g for R then $W = g^{-1}(\mathbb{D} \setminus W)$ had to be a G_δ-set too. A G_δ-subset of a topologically complete space is topologically complete too ([He], Satz 3.5.6), and a topologically complete space is of second category by Baire's Category Theorem ([He], Theorem 3.4.3, Folgerung 3.5.2). But W is obviously not of second category. Thus, there is no continuous selector for R.

This implies that R is not relatively continuous since we can choose $id : \mathbb{D} \longrightarrow \mathbb{D}$ as admissible representation of $\mathbb{D}$ by Proposition 4.3. □

Example 7.2 *The relation*

$$S := (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Q} \cup \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$$

is continuous, but it does not have a continuous selector and it is not relatively continuous.

Proof. The relation S is continuous. That there is no selector for S can be proved in the same way as for R.

In order to show that S is not relatively continuous we first consider the normed Cauchy representation $\delta :\subseteq \mathbb{D} \longrightarrow \mathbb{R}$. We need three properties:

(1) δ is admissible,

(2) $dom(\delta)$ is a G_δ-set,

(3) for any $q \in \mathbb{Q}$ the set $\delta^{-1}(q)$ is nowhere dense in $M := \delta^{-1}(\mathbb{Q})$.

The first two properties can be easily checked. In order to prove property (3) we use the obvious fact that for any non-empty open subset $B \subseteq dom(\delta)$ the interior of $\delta(B)$ is non-empty.

Proof of (3): Fix a $q \in \mathbb{Q}$. Since δ is continuous the set $\delta^{-1}(q)$ is closed in $dom(\delta)$ and hence in M too. Let us assume that $\delta^{-1}(q)$ is not nowhere dense in M. Then there is a non-empty set $A \subseteq \delta^{-1}(q)$ which is open in M. Fix an open subset $B \subseteq dom(\delta)$ with $B \cap M = A$. The interior of $\delta(B)$ is non-empty (see above). Thus, the interior of $\delta(B) \cap \mathbb{Q} = \delta(A)$ in $\mathbb{Q}$ is non-empty. But this is impossible because $\delta(A) = \{q\}$. □

Now suppose that S is relatively continuous, i.e. S is (δ, η)-continuous for an admissible representation $\eta :\subseteq \mathbb{D} \longrightarrow \mathbb{R}$. There is a continuous function $F :\subseteq \mathbb{D} \longrightarrow \mathbb{D}$ such that $(\forall p \in dom(\delta))\, \eta F(p) \in R \circ \delta(p)$ where w.l.o.g. $dom(F) = dom(\delta)$. The function $\Delta := \eta \circ F$ is continuous since η is continuous. Thus the set $M := \Delta^{-1}(\mathbb{R} \backslash \mathbb{Q})$ is a G_δ-set in $dom(\delta)$ because $\mathbb{R} \backslash \mathbb{Q}$ is a G_δ-set. Since a subset of a topologically complete set is topologically complete iff it is a G_δ-set ([He], Satz 3.5.6), $dom(\delta)$ and M are topologically complete. By Baire's Category Theorem a topologically complete space is of second category.

On the other hand $M = \delta^{-1}(\mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \delta^{-1}(q)$. Hence by (3) M is the countable union of sets that are nowhere dense in M, i.e. M is not of second category. Contradiction! □

Nevertheless sometimes one might wish to consider arbitrary relations and not just relations with closed images. We show that in certain cases for relatively continuous relations it does not matter whether one considers arbitrary relations or just relations with compact images.

Definition 7.3 A representation $\delta :\subseteq \mathbb{D} \longrightarrow X$ of a set X *has compact fibers* iff $\delta^{-1}\{x\}$ is compact for any $x \in X$

Lemma 7.4 *Let X be a space which has an admissible representation $\delta :\subseteq \mathbb{D} \longrightarrow X$ with compact fibers, and let Y be T_2-space with countable base. Then any relatively continuous relation $R \subseteq X \times Y$ has a relatively continuous restriction R' with compact images (and with $dom(R) = dom(R')$).*

Proof. Fix an admissible representation $\eta :\subseteq \mathbb{D} \longrightarrow Y$. Since R is relatively continuous there is a continuous function $F :\subseteq \mathbb{D} \longrightarrow \mathbb{D}$ such that $(\forall p \in dom(R\delta))\, \eta F(p) \in R\delta(p)$ where w.l.o.g. $dom(F) = dom(R\delta)$. The relation

$$R_F := \{(\delta(p), \eta F(p)) | p \in dom(\eta F)\}$$

is relatively continuous and fulfills

$$R_F \subseteq R \text{ and } dom(R_F) = dom(R).$$

For any $x \in X$ the set $R_F(x) = \eta F(\delta^{-1}(x))$ is compact because δ has compact fibers and ηF is a continuous map between T_2-spaces. $\square$

At the end of section 6 we mentioned that an analogous result to Corollary 5.5 holds true for arbitrary — not necessarily complete — separable metric spaces Y if one considers only relations with compact images. Combined with the last lemma this implies:

Proposition 7.5 *Let X be space which has an admissible representation $\delta :\subseteq \mathbb{D} \longrightarrow X$ with compact fibers, and let Y be a separable metric space. Then for any relation $R \subseteq X \times Y$ the following two conditions are equivalent:*

(1) R is relatively continuous.

(2) R has a continuous restriction with compact images.

Are there any interesting spaces X which have an admissible representation $\delta :\subseteq \mathbb{D} \longrightarrow X$ with compact fibers? Indeed there are. Matthias Schröder has shown (private communication) that each separable metric space X has an admissible representation with compact fibers.

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