# Computability on Asymmetric Spaces

Habilitationsschrift

dem Fachbereich Informatik
der FernUniversität - Gesamthochschule in Hagen

vorgelegt von

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im Dezember 2002

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# Chapter 1

# Introduction

The mathematical sciences particularly exhibit order, symmetry, and limitation;
and these are the greatest forms of the beautiful.
(Aristotle, Metaphysica, 3–1078b)

As in other fields of mathematical sciences, symmetry plays an essential role in computer science. However, in many applications one has to deal with incomplete information and this often implies that asymmetry is unavoidable. This is the case, for instance, if we can obtain only lower or upper bounds on real numbers and not both, if either positive or negative information on sets is available or if we can approximate functions just from above or below. If we want to deal with these situations appropriately, then we have to learn how to handle asymmetry. In this respect, topology turns out to be the main mathematical tool (see [Smy92]).

In this work we will approach the aim to control asymmetry from the point of view of computable analysis which is the Turing machine based theory of computable points, subsets and functions on Euclidean space and other topological spaces (see [Wei00]). It is a general phenomenon that any careful investigation in computable analysis goes hand in hand with a topological exploration of the underlying spaces. This is because it first has to be clarified how infinite objects are to be approximated before one can effectively compute with them. Throughout the whole investigation we will use the three mentioned settings of computable points (with lower or upper bounds), computable sets (with positive or negative information) and of computable functions (with approximations from below or above) as instructive and guiding examples.

One of the main goals of this work is to derive *data structures* for topological spaces which are used to handle objects with respect to asymmetric information. Here data structures are just sets (i.e. data types) which are endowed

| set $X$ | points | quasi-metric $d : X \times X \to \mathbb{R}$ |
|---|---|---|
| $\mathbb{R}$ | real numbers | $(x, y) \mapsto x \dot{-} y$ |
| $\mathcal{K}(\mathbb{R}^n)$ | compact subsets | $(A, B) \mapsto \sup_{a \in A} \inf_{b \in B} d(a, b)$ |
| $\mathcal{USC}(\mathbb{R}^n)$ | upper semi-continuous functions | $(f, g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \left\| \frac{f \dot{-} g}{1+(f \dot{-} g)} \right\|_{[-i,i]^n}$ |
| $\mathcal{A}(\mathbb{R}^n)$ | closed subsets | $(A, B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \|d_B \dot{-} d_A\|_{[-i,i]^n}$ |

Table 1.1: Some quasi-metric spaces

with certain initial operations. If a topological space comes as it is (without any additional algebraic or topological structure), then it is very hard to imagine how canonical initial operations of data structures should be gained[1]. In the symmetric case, metric spaces have been proved to be a powerful tool to obtain canonical initial operations. Therefore, in the asymmetric case it is natural to consider a similar approach with quasi-metric spaces which are, roughly speaking, metric spaces without the symmetry axiom. And actually, we will follow this line and show that it leads to a solution of our problem. In Table 1.1 some quasi-metrics are given for point, hyper and function spaces.

Of course, it is not just our goal to construct *some* data structures for certain asymmetric spaces, but we want these data structures to reflect the computational structure of the space. In other words, these structures have to be *sound* as well as *complete* in the sense that, on the one hand, they are implementable and, on the other hand, they keep the whole expressive power of the computational structure. It is the theory of *perfect topological structures* which we will apply to fulfill these constraints (this theory has been presented in the author's thesis, see [Bra99a, Bra02a]).

It turns out that any such data structure which is sound as well as complete has *synthetic* as well as *analytic* operations. While synthetic operations are present to guarantee that sufficiently many computable operations can be generated, analytic operations guarantee the usability of the results of computations (typically, by transferring them to simpler data structures). In case of metric structures the role of the synthetic operation is played by the limit operation and the role of the analytic operation by the metric itself. In case of quasi-metric structures the analytic part is represented by the quasi-metric and the synthetic part by a corresponding infimum operation (which is induced by the partial order that is canonically associated with any quasi-metric). In Table 1.2 we continue our example of quasi-metric spaces with the corresponding

[1]In the real number case, for instance, the arithmetical operations are canonical candidates for initial operations, but they are induced by the algebraic structure of the reals.

| set | partial order | infimum | supremum |
| --- | --- | --- | --- |
| $\mathbb{R}$ | $\leq$ | $(x_n) \mapsto \inf_{n\in\mathbb{N}} x_n$ | $(x_n) \mapsto \sup_{n\in\mathbb{N}} x_n$ |
| $\mathcal{K}(\mathbb{R}^n)$ | $\subseteq$ | $(K_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} K_n$ | $(K_n)_{n\in\mathbb{N}} \mapsto \overline{\bigcup_{n=0}^{\infty} K_n}$ |
| $\mathcal{USC}(\mathbb{R}^n)$ | $\leq$ | $(f_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} f_n$ | $(f_n)_{n\in\mathbb{N}} \mapsto \sup_{n\in\mathbb{N}} f_n$ |
| $\mathcal{A}(\mathbb{R}^n)$ | $\subseteq$ | $(A_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} A_n$ | $(A_n)_{n\in\mathbb{N}} \mapsto \overline{\bigcup_{n=0}^{\infty} A_n}$ |

Table 1.2: Some quasi-metric spaces with their associated partial orders

partial order, the infimum and the supremum operation.

Altogether, this approach leads to very natural data structures for many quasi-metric spaces. Let us illustrate this in case of the hyperspace $\mathcal{K}(\mathbb{R}^n)$ of non-empty compact subsets $K \subseteq \mathbb{R}^n$. As far as symmetric information is concerned, it is known by results of [Bra99a] that the structure

$$\mathcal{K}(\mathbf{R}^n) := \mathbf{R} \oplus (\mathcal{K}(\mathbb{R}^n); \{x\}, A, A \cup B, d_{\mathcal{K}}, \mathrm{Lim})$$

is a perfect topological structure. This is a many-sorted structure which consists of the ordinary structure of the real numbers $\mathbf{R}$ plus a structure for the hyperspace of compact subsets. Here, the initial operations are the injection of a single point into a singleton, the identity, the ordinary set union, the Hausdorff metric and a corresponding limit operation (see the tables in the appendix for further details). Now, from the theory of perfect structures one can conclude that this structure reflects the computability theory on $\mathcal{K}(\mathbb{R}^n)$ very well: programs which are build by operations of this structure precisely represent the computable operations on $\mathcal{K}(\mathbb{R}^n)$. As we will see in Chapter 4, we can analogously handle the hyperspace of compact subsets in the asymmetric case. For instance, we can use the operations $(K_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} K_n$ (from Table 1.2) and $(A, B) \mapsto \sup_{a\in A} \inf_{b\in B} d(a, b)$ (from Table 1.1) to define a structure for the space of compact subsets, with respect to negative information (for details see Chapter 4). Thus, the data structures for hyperspaces are endowed with very natural operations such as union, intersection and maximal distance from one set to another. Using the conjugate quasi-metric one obtains similar results in the case of positive information. Figure 1.1 gives an overview of some symmetric and asymmetric spaces which we will investigate. These spaces are given together with the names of the corresponding computable points and the underlying topologies. The data structures for these spaces are listed in tables in the appendix.

Figure 1.1 also suggests that the quasi-metric approach explains a striking observation in computable analysis: many sets admit three different natural representations and one of these often is the join of the other two. Now we can

| | | |
|---|---|---|
| $\mathcal{A}_<(\mathbb{R}^n)$<br>• r.e. closed sets<br>• lower Fell topology | $\mathcal{A}(\mathbb{R}^n)$<br>• recursive closed sets<br>• Fell topology | $\mathcal{A}_>(\mathbb{R}^n)$<br>• co-r.e. closed sets<br>• upper Fell topology |
| $\mathcal{LSC}(\mathbb{R}^n)$<br>• lower semi-comp. fct.<br>• lower compact open top. | $\mathcal{C}(\mathbb{R}^n)$<br>• computable functions<br>• compact open topology | $\mathcal{USC}(\mathbb{R}^n)$<br>• upper semi-comp. fct.<br>• upper compact open top. |
| $\mathcal{K}_<(\mathbb{R}^n)$<br>• r.e. compact sets<br>• lower Vietoris topology | $\mathcal{K}(\mathbb{R}^n)$<br>• recursive compact sets<br>• Vietoris topology | $\mathcal{K}_>(\mathbb{R}^n)$<br>• co-r.e. compact sets<br>• upper Vietoris topology |
| $\mathbb{R}_<$<br>• left computable reals<br>• lower Euclidean top. | $\mathbb{R}$<br>• computable reals<br>• Euclidean topology | $\mathbb{R}_>$<br>• right computable reals<br>• upper Euclidean top. |
| conjugate quasi-metric | metric | quasi-metric |

Figure 1.1: Metric and quasi-metric spaces

clearly see that for typical spaces all three structures are induced by only one characteristic map and its conjugation and symmetrization: a quasi-metric. While for some spaces it was known that the symmetric representation can be induced by a metric, the asymmetric cases have always been treated by ad-hoc techniques. The quasi-metric approach now offers a unified canonical method to handle all these cases systematically.

We close this introduction with a survey of the organization of the following chapters. Chapter 2 is dedicated to a purely topological investigation of quasi-metric spaces. In formal analogy to separable metric spaces we introduce the concept of a generated quasi-metric space. In a corresponding way as each point of a separable metric space can be represented as the limit of a sequence in some countable dense subset, each point of a generated quasi-metric space can be considered as the infimum of a sequence in the generating set (with respect to the partial order induced by the quasi-metric). Typically, the generating subset can be chosen such that it is itself a separable metric space (with respect to the metric induced by the quasi-metric). This concept enables a "countable access" to the points of the quasi-metric space. We prove that certain important hyper and function spaces can be naturally considered as generated quasi-metric spaces and we investigate further topological properties of these spaces which are helpful in following chapters. Chapter 2 is mainly based on an extended version of the paper [Bra03].

Chapter 3 continues the study of quasi-metric spaces from the computational point of view. In analogy to recursive metric spaces we introduce the concept of a semi-recursive quasi-metric space. It turns out that these spaces have similar properties as recursive metric spaces: any such space admits a canonical representation, the so-called Dedekind representation. This representation is admissible (i.e. topologically well-behaved) and its equivalence class can naturally be characterized by the basic operations of the quasi-metric space (the quasi-metric and the infimum operation). Finally, we prove that our examples of hyper and function spaces are semi-recursive quasi-metric spaces. Chapter 3 is mainly based on an extended version of the paper [Braar].

In Chapter 4 we extend the theory of perfect topological structures to quasi-metric structures. Based on the presentation in [Bra02a] (which in turn is based on the author's thesis [Bra99a]) we give a brief introduction into the theory of perfect topological structures in the first half of Chapter 4. In the second half quasi-metric structures are investigated. In particular we derive perfect quasi-metric structures for all our examples of point, hyper and function spaces. The material on quasi-metric structures has not been published before.

Chapter 5 contains some additional material which in particular shows that the Dedekind representations of our hyper and function spaces are actually equivalent to certain representations studied in computable analysis. Moreover, we discuss the possibility of effective quasi-metrizability. In particular, we study a canonical construction of a quasi-metric for second-countable $T_0$–spaces. The content of this chapter has not been published before.

In the Conclusion we try to make the underlying theses of the work explicit and we mention a number of open questions and directions of future research. At this point we also mention that the apparent relation of the material to domain theory has not been the object of our investigation.

We close this introduction with a list of those references which cover essential parts of Chapters 2 and 3 as well as the main part of the first half of Chapter 4:

[Bra03] Vasco Brattka. Generated quasi-metric hyper and function spaces. *Topology and its Applications*, 127:355–373, 2003.

[Braar] Vasco Brattka. Recursive quasi-metric spaces. *Theoretical Computer Science*, to appear.

[Bra02a] Vasco Brattka. Computability over topological structures. In S. Barry Cooper and Sergey Goncharov, editors, *Computability and Models*, pages 93–136. Kluwer Academic Publishers, Dordrecht, 2002.

**Acknowledgements**

This work has been produced while I was a research fellow at the Computer Science Department of the University of Hagen and I am indebted to Klaus Weihrauch, Matthias Schröder and Peter Hertling for their support; their own work has influenced my project in many aspects. A fruitful discussion with Hans-Peter A. Künzi on quasi-metric spaces increased my interest in asymmetric topologies. Several remarks of anonymous referees (on earlier versions of published parts of the material) helped to improve the presentation. Finally, I would like to acknowledge that during the last two years this project has been supported by a grant of the Deutsche Forschungsgemeinschaft (DFG grant BR 1807/4-1).

# Chapter 2

# Quasi-Metric Spaces

## 2.1 Introduction

In this chapter we will discuss some purely topological properties of quasi-metric spaces. In Section 2 we will start with the definition of the notion of a *generated quasi-metric space*. In a corresponding way as each point of a metric space with a dense subset can be represented as the limit of a sequence in the dense subset, each point of an upper generated quasi-metric space can be considered as the infimum of a sequence in the generating set (with respect to the partial order induced by the quasi-metric). In case of metric spaces, the dense subset can often be chosen to be countable. In contrast to that, in case of quasi-metric spaces the generating subset will typically be non-countable. Nevertheless, the "countable access" to the space will be guaranteed since in all our applications the generating set will be a separable metric space itself (with respect to the symmetric distance induced by the quasi-metric).

In Section 3 we will discuss some continuity properties of quasi-metrics. In Section 4 we will introduce the notion of *strong density*, which is helpful to prove that certain quasi-metric spaces are upper generated by certain subspaces. Moreover, in the next chapter this property guarantees that the corresponding spaces have some nice computability properties.

In Section 4, 5, and 6 we investigate some generated quasi-metric hyper and function spaces. More precisely, we will endow the hyperspace of non-empty compact subsets $\mathcal{K}(X)$ and the hyperspace on non-empty closed subsets $\mathcal{A}(X)$ of certain metric spaces $X$ with a quasi-metric. As function spaces we consider the spaces $\mathcal{USC}(X)$ and $\mathcal{LSC}(X)$ of upper and lower semi-continuous functions $f : X \to \mathbb{R}$. We show that all these spaces are generated quasi-metric spaces in a certain sense and that the corresponding topologies are well-known topologies. Finally, we prove that the generating subsets are strongly dense within these spaces and that the quasi-metrics fulfill some natural continuity properties. Altogether, the essential topological properties of these spaces, used in the next chapters, are clarified.

## 2.2 Generated quasi-metric spaces

In this section we will introduce the concept of a generated quasi-metric space. Quasi-metric spaces are roughly speaking like metric spaces without symmetry. There exist several slightly different definitions in the literature and we will follow the definition of Smyth [Smy92]. Other applications of quasi-metrics to theoretical computer science can be found e.g. in [Sün95, HS99]. For a survey on the history of asymmetric spaces cf. Künzi [Kün01]. Most results of this section are well-known (see [Smy92]) and included for completeness.

**Definition 2.2.1 (Quasi-metric spaces)** We will call $(X, d)$ a *quasi-metric space*, if $d : X \times X \to \mathbb{R}$ is a non-negative function, called *quasi-metric*, such that

(1) $d(x, x) = 0$,

(2) $d(x, y) = d(y, x) = 0 \Longrightarrow x = y$,

(3) $d(x, y) \leq d(x, z) + d(z, y)$

hold for all $x, y, z \in X$.

With each quasi-metric space $(X, d)$ we can associate the *conjugate quasi-metric space* $(X, \overline{d})$ with the quasi-metric

$$\overline{d} : X \times X \to \mathbb{R}, (x, y) \mapsto d(y, x).$$

Each quasi-metric space $(X, d)$ induces a *lower topology* $\tau_<$ with the basic open sets $B_<(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$, and an *upper topology* $\tau_>$ with the basic open sets $B_>(x, \varepsilon) := \{y \in X : d(y, x) < \varepsilon\}$, for all $x \in X, \varepsilon > 0$. We prove that these sets actually define bases.

**Lemma 2.2.2** *Let $(X, d)$ be a quasi-metric space. Then*

*(1) $\{B_<(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base of a topology $\tau_<$,*

*(2) $\{B_>(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base of a topology $\tau_>$.*

**Proof.** We only prove (1) since (2) can be proved analogously. Obviously, $X = \bigcup_{x \in X} B_<(x, 1)$. Now let $x, x' \in X$, $\varepsilon, \varepsilon' > 0$ and $y \in B_<(x, \varepsilon) \cap B_<(x', \varepsilon')$. Then $\delta := \min\{\varepsilon - d(x, y), \varepsilon' - d(x', y)\} > 0$ and $d(y, z) < \delta$ for $z \in X$ implies

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon$$

and analogously $d(x', z) < \varepsilon'$, i.e. $z \in B_<(x, \varepsilon) \cap B_<(x', \varepsilon')$. Thus, we obtain $y \in B_<(y, \delta) \subseteq B_<(x, \varepsilon) \cap B_<(x', \varepsilon')$ which implies the claim. □

Obviously, the lower topology, induced by a quasi-metric $d$, coincides with the upper topology, induced by the conjugate quasi-metric $\overline{d}$, and vice versa. In the following we will usually fix a quasi-metric $d$ and consider its lower and upper topology. If $(X, d)$ is some fixed quasi-metric space, then we will write $X_<$ and $X_>$ for the space $X$ endowed with the lower and upper topology, respectively.

**Example 2.2.3 (Real numbers)** *We define the* truncated difference *by*

$$\dot{-} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto \begin{cases} x - y & \text{if} \quad x > y \\ 0 & \text{else} \end{cases}.$$

*Then $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto x \dot{-} y$ is a quasi-metric on $\mathbb{R}$. Here, the triangle inequality holds since for all $x, y, z \in \mathbb{R}$ we obtain*

$$\begin{aligned} x \dot{-} y &= \max\{0, x - y\} = \max\{0, (x - z) + (z - y)\} \\ &\leq \max\{0, x - z\} + \max\{0, z - y\} = (x \dot{-} z) + (z \dot{-} y). \end{aligned}$$

By $\mathbb{R}_<$ and $\mathbb{R}_>$ we denote the real numbers endowed with the corresponding lower and upper topology, respectively. It is easy to see that the lower topology is generated by the basic open sets $(q, \infty)$ with $q \in \mathbb{Q}$ and the upper topology is generated by the basic open sets $(-\infty, q)$ with $q \in \mathbb{Q}$. Quasi-metrics are continuous in the following sense.

**Lemma 2.2.4 (Continuity of quasi-metrics)** *Let $(X, d)$ be a quasi-metric space. Then*

*(1) $d : X_> \times X_< \to \mathbb{R}_>$ is continuous,*

*(2) $d : X_< \times X_> \to \mathbb{R}_<$ is continuous.*

**Proof.**

(1) Let $r \in \mathbb{R}$ and $(x, y) \in d^{-1}(-\infty, r)$. Then $d(x, y) < r$ and $\varepsilon := \frac{r - d(x,y)}{2} > 0$. Let $x' \in B_>(x, \varepsilon)$ and $y' \in B_<(y, \varepsilon)$. Then

$$d(x', y') \leq d(x', x) + d(x, y) + d(y, y') < d(x, y) + 2\varepsilon = r.$$

Thus, $d(x', y') < r$ and $B_>(x, \varepsilon) \times B_<(y, \varepsilon) \subseteq d^{-1}(-\infty, r)$.

(2) Let $r \in \mathbb{R}$ and $(x, y) \in d^{-1}(r, \infty)$. Then $d(x, y) > r$ and $\varepsilon := \frac{d(x,y)-r}{2} > 0$. Let $x' \in B_<(x, \varepsilon)$ and $y' \in B_>(y, \varepsilon)$. Then

$$d(x', y') \geq d(x, y) - d(x, x') - d(y', y) > d(x, y) - 2\varepsilon = r.$$

Thus, $d(x', y') > r$ and $B_<(x, \varepsilon) \times B_>(y, \varepsilon) \subseteq d^{-1}(r, \infty)$.

□

With each quasi-metric space $(X, d)$ we can associate a metric space $(X, d_\star)$ with the metric

$$d_\star : X \times X \to \mathbb{R}, (x, y) \mapsto \max\{d(x, y), \overline{d}(x, y)\}.$$

If $d$ itself is a metric, then obviously $d_\star = d$. We denote the open balls with respect to $d_\star$ by $B(x, \varepsilon) := \{y \in X : d_\star(x, y) < \varepsilon\}$. We prove that the associated metric topology is the join topology of the lower and the upper topology. If $\tau, \tau'$ are topologies for the same set $X$, then we denote by

$$\tau \sqcap \tau' := \{U \cap V : U \in \tau, V \in \tau'\}$$

the *join topology* of $\tau$ and $\tau'$.

**Lemma 2.2.5** *Let $(X, d)$ be a quasi-metric space. The topology $\tau$ induced by $d_\star$ is the join topology of the lower and upper topology, i.e. $\tau = \tau_< \sqcap \tau_>$.*

**Proof.** On the one hand, we obtain $B_<(x, \varepsilon) = \bigcup_{y \in B_<(x,\varepsilon)} B(y, \varepsilon - d(x, y))$ and $B_>(x, \varepsilon) = \bigcup_{y \in B_>(x,\varepsilon)} B(y, \varepsilon - d(y, x))$ and thus we can conclude $\tau_< \subseteq \tau$ and $\tau_> \subseteq \tau$ and hence $\tau_< \sqcap \tau_> \subseteq \tau$. On the other hand, we obtain

$$\begin{aligned} B(x, \varepsilon) &= \{y \in X : d_\star(x, y) < \varepsilon\} \\ &= \{y \in X : d(x, y) < \varepsilon \text{ and } d(y, x) < \varepsilon\} \\ &= B_<(x, \varepsilon) \cap B_>(x, \varepsilon) \end{aligned}$$

and thus $\tau \subseteq \tau_< \sqcap \tau_>$. □

With each quasi-metric space $(X, d)$ we can associate a *partial order* $\sqsubseteq$ which is a subset of $X \times X$, defined by

$$x \sqsubseteq y :\iff d(x, y) = 0$$

(i.e. $\sqsubseteq$ is reflexive, transitive and anti-symmetric). By inf and sup we denote the *infimum* and *supremum* of the partially ordered space $(X, \sqsubseteq)$, respectively. A sequence $(x_n)_{n \in \mathbb{N}}$, with $x_i \sqsubseteq x_{i+1}$ for all $i$ will be called an *increasing chain*, and analogously, a *decreasing chain*, if $x_{i+1} \sqsubseteq x_i$. We will say that a partially ordered space $(X, \sqsubseteq)$ is *inf-complete*, if each decreasing chain has an infimum, and analogously, we will say that it is *sup-complete*, if each increasing chain has a supremum. Furthermore, we will say that $(X, \sqsubseteq)$ is an *inf-semi-lattice* if each pair in $X$ has an infimum, we will say that it is a *sup-semi-lattice*, if each pair in $X$ has a supremum and we will say that it is a *lattice* if both conditions hold. In general, the partially ordered space $(X, \sqsubseteq)$ induced by a quasi-metric is neither complete nor a lattice. By the triangle inequality it is easy to see, that each quasi-metric is isotone in the following way:

$$x \sqsubseteq y \Longrightarrow d(x,z) \leq d(y,z) \text{ and } d(z,y) \leq d(z,x)$$

for all $x, y, z \in X$. As a consequence, in case $\inf_{n\in\mathbb{N}} x_n$ or $\sup_{n\in\mathbb{N}} x_n$ exist,

$$d\left(\inf_{n\in\mathbb{N}} x_n, y\right) \leq \inf_{n\in\mathbb{N}} d(x_n, y) \quad \text{or} \quad d\left(\sup_{n\in\mathbb{N}} x_n, y\right) \geq \sup_{n\in\mathbb{N}} d(x_n, y)$$

respectively, for all sequences $(x_n)_{n\in\mathbb{N}}$ and $y$ in $X$. In general, equality does not hold, neither for chains $(x_n)_{n\in\mathbb{N}}$ nor for finite sequences.

**Example 2.2.6** *Let* $X := \{-1\} \cup (0, 1]$ *and* $d : X \times X \to \mathbb{R}, (x, y) \mapsto x \dot{-} y$. *Then* $(X, d)$ *is a quasi-metric space and the partial order induced by* $d$ *is the usual order* $\leq$. *Thus, by* $x_n := 2^{-n}$ *a decreasing chain with* $x := \inf_{n\in\mathbb{N}} x_n = -1$ *is defined and* $d(\inf_{n\in\mathbb{N}} x_n, x) = 0 < 1 = \inf_{n\in\mathbb{N}} d(x_n, x)$.

In the next definition we introduce a generation property which is a counterpart to the notion of a dense subset of a metric space. In the same sense as each point of a separable metric space can be obtained as a limit of a sequence in a dense subset, we can obtain each point of an upper generated quasi-metric space as infimum of a sequence in a suitable subset.

**Definition 2.2.7 (Generated quasi-metric spaces)** A tuple $(X, Y, d)$ is called *upper generated quasi-metric space*, if $(X, d)$ is a quasi-metric space with a subset $Y \subseteq X$, such that each $x \in X$ is the infimum of a sequence of points in $Y$. Analogously, we define *lower generated quasi-metric spaces* with suprema instead of infima.

Thus, $(X, Y, d)$ is an upper generated quasi-metric space, if and only if $(X, d)$ is a quasi-metric space, $Y \subseteq X$ and

$$\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X, (x_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} x_n$$

is surjective. A corresponding property holds for lower generated spaces with sup instead of inf. Here and in the following the inclusion symbol "$\subseteq$" indicates that a function might be partial. Typically, we will endow the subset $Y$ with the associated metric $d_\star$ and in all our applications $(Y, d_\star)$ will be separable. A simple standard example of an upper and lower generated quasi-metric space is the following.

**Example 2.2.8** *The space $(\mathbb{R}, \mathbb{R}, d)$ with $d(x, y) := x \dot{-} y$ is an upper as well as a lower generated quasi-metric space, the partial order induced by d is the usual order $\leq$, the associated metric $d_\star$ is the Euclidean metric.*

## 2.3 Continuity properties of quasi-metrics

In this section we want to study some continuity properties of generated quasi-metric spaces. As we have seen in the previous sections, a quasi-metric is continuous with respect to its lower and upper topology in a certain sense. Unfortunately, it is not continuous with respect to the induced partial order $\sqsubseteq$ in general (as Example 2.2.6 shows). For generated quasi-metric spaces we will introduce a special kind of continuity with respect to $\sqsubseteq$.

**Definition 2.3.1 (Continuity from above)** An upper generated quasi-metric space $(X, Y, d)$ is called *continuous from above*, if

$$d\left(\inf_{n\in\mathbb{N}} y_n, y\right) = \inf_{n\in\mathbb{N}} d(y_n, y)$$

holds for all $y \in Y$ and all decreasing chains $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ such that $\inf_{n\in\mathbb{N}} y_n$ exists in $X$.

It is quite convenient to have another notion which expresses a closely related property.

**Definition 2.3.2 (Consistent quasi-metric spaces)** Let $(X, Y, d)$ be an upper generated quasi-metric space with a topology $\tau$ on $X$. Then $(X, Y, d)$ is called *consistent from above with respect to $\tau$*, if each decreasing chain $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ such that $x := \inf_{n\in\mathbb{N}} y_n$ exists in $X$, is convergent to $x$ with respect to $\tau$. Analogously, the property *consistent from below* can be defined for lower generated quasi-metric spaces.

If a quasi-metric space is consistent from above with respect to the upper topology, then the dual quasi-metric space is consistent from below with respect to the lower topology and vice versa. A corresponding statement does not hold for arbitrary topologies $\tau$, especially not for the weak upper topology, which is defined as follows.

**Definition 2.3.3 (Weak upper topology)** Let $(X, Y, d)$ be an upper generated quasi-metric space. Then the *weak upper topology* is the topology on $X$ which is induced by the open balls $B_>(y, \varepsilon)$ with $y \in Y$ and $\varepsilon > 0$.

Of course, one could define a corresponding concept of a weak lower topology, but we will not use this idea in the following. It is obvious that in case of $Y = X$ the weak upper topology coincides with the upper topology on $X$. But in general the weak upper topology is a proper subset of the upper topology. Moreover, it is easy to see that the weak upper topology admits a countable base, if $(Y, d_\star)$ is separable. With the help of the weak upper topology we can characterize continuity from above as stated in the following proposition. If $Y$ is endowed with a topology, then we will always assume that $Y^{\mathbb{N}}$ is endowed with the corresponding product topology.

**Proposition 2.3.4 (Continuity from above)** *Let $(X, Y, d)$ be an upper generated quasi-metric space and let $Y$ be endowed with the metric topology induced by $d_\star$. Consider the following conditions:*

*(1) $(X, Y, d)$ is continuous from above,*

*(2) $(X, Y, d)$ is consistent from above w.r.t. the weak upper topology on $X$,*

*(3) $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X$ is continuous w.r.t. the weak upper topology on $X$.*

*Then (1) and (2) are equivalent, (3) implies (2) and if, additionally,*

$$\sqcap : Y \times Y \to Y, (y, y') \mapsto \inf\{y, y'\}$$

*is total and continuous, then (2) also implies (3).*

**Proof.** "(1)$\Longrightarrow$(2)" Let $(X, Y, d)$ be continuous from above and let $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ be a decreasing chain such that $x := \inf_{n\in\mathbb{N}} y_n$ exists in $X$. Let $y \in Y$, $\varepsilon > 0$ such that $x \in B_>(y, \varepsilon)$, i.e. $d(x, y) < \varepsilon$. Since $(X, Y, d)$ is continuous from above, it follows $\inf_{n\in\mathbb{N}} d(y_n, y) = d(x, y) < \varepsilon$. Thus, there is a $k \in \mathbb{N}$ such that $d(y_k, y) < \varepsilon$ and since $(y_n)_{n\in\mathbb{N}}$ is a decreasing chain and $d$ is isotone in the first component, it follows $d(y_n, y) \le d(y_k, y) < \varepsilon$, i.e. $y_n \in B_>(y, \varepsilon)$ for all $n \ge k$. Thus, $(y_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the weak upper topology.

"(2)$\Longrightarrow$(1)" Let $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ be a decreasing chain such that $x := \inf_{n\in\mathbb{N}} y_n$ exists in $X$ and let $y \in Y$. We have to prove $d(x, y) = \inf_{n\in\mathbb{N}} d(y_n, y)$. Since $d$ is isotone in the first component, we obtain $d(x, y) \le \inf_{n\in\mathbb{N}} d(y_n, y) =: r$. Let us assume $d(x, y) < r$, i.e. $x \in B_>(y, r)$. Since $(y_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the weak upper topology, there is some $k \in \mathbb{N}$ such that $y_n \in B_>(y, r)$

for all $n \geq k$, i.e. $d(y_n, y) < r = \inf_{n\in\mathbb{N}} d(y_n, y)$ for all $n \geq k$. Contradiction! Thus, $d(x, y) = r = \inf_{n\in\mathbb{N}} d(y_n, y)$.

"(3)$\Longrightarrow$(2)" Let $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ be a decreasing chain such that $x := \inf_{n\in\mathbb{N}} y_n$ exists in $X$ and let $y \in Y$ and $\varepsilon > 0$ be such that $x \in B_>(y, \varepsilon)$. Then $(y_n)_{n\in\mathbb{N}} \in \mathrm{Inf}^{-1}B_>(y, \varepsilon)$. Since Inf is continuous, it follows that there is some $k \in \mathbb{N}$ and some $\delta > 0$ such that

$$(y_n)_{n\in\mathbb{N}} \in U := B(y_0, \delta) \times ... \times B(y_k, \delta) \times Y \times Y \times ... \subseteq \mathrm{Inf}^{-1}B_>(y, \varepsilon).$$

Let $(z_n)_{n\in\mathbb{N}}$ be defined by $z_i := y_i$ for $i = 0, ..., k$ and $z_i := y_k$ for all $i > k$. Then $(z_n)_{n\in\mathbb{N}} \in U$ and since $(y_n)_{n\in\mathbb{N}}$ is decreasing, we obtain $y_k = \inf_{n\in\mathbb{N}} z_n$ and since $d$ is isotone in the first component, $d(y_n, y) \leq d(y_k, y) = d(\inf_{n\in\mathbb{N}} z_n, y) < \varepsilon$ for all $n \geq k$, i.e. $y_n \in B_>(y, \varepsilon)$ for all $n \geq k$. Thus, $(y_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the weak upper topology.

"(1)$\Longrightarrow$(3)" Let $\sqcap$ be total and continuous, let $(X, Y, d)$ be continuous from above and let $y \in Y$ and $\varepsilon > 0$. Define $F : Y^{\mathbb{N}} \to \mathbb{R}_>^{\mathbb{N}}$ by

$$F((y_n)_{n\in\mathbb{N}}) := (d(\inf\{y_0, ..., y_k\}, y))_{k\in\mathbb{N}}$$

for all $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$. By Lemma 2.2.4 (1) the function $d|_{Y\times Y} : Y \times Y \to \mathbb{R}_>$ is continuous and thus it follows that $F$ is continuous since $\sqcap$ is continuous. Moreover, totality of $\sqcap$ implies that $\inf\{y_0, ..., y_k\}$ exists for all $y_0, ..., y_k \in Y$. Since $(X, Y, d)$ is continuous from above, we obtain

$$\begin{aligned}
\mathrm{Inf}^{-1}B_>(y, \varepsilon) &= \left\{(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}} : d\left(\inf_{n\in\mathbb{N}} y_n, y\right) < \varepsilon\right\} \\
&= \left\{(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}} : d\left(\inf_{k\in\mathbb{N}} \inf\{y_0, ..., y_k\}, y\right) < \varepsilon\right\} \\
&= \left\{(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}} : \inf_{k\in\mathbb{N}} d(\inf\{y_0, ..., y_k\}, y) < \varepsilon\right\} \cap \mathrm{dom}(\mathrm{Inf}) \\
&= (\mathrm{Inf}_{\mathbb{R}_>} \circ F)^{-1}(-\infty, \varepsilon) \cap \mathrm{dom}(\mathrm{Inf}).
\end{aligned}$$

Since $\mathrm{Inf}_{\mathbb{R}_>} :\subseteq \mathbb{R}_>^{\mathbb{N}} \to \mathbb{R}_>$ is continuous and $F : Y^{\mathbb{N}} \to \mathbb{R}_>^{\mathbb{N}}$ is continuous, it follows that Inf is continuous. □

## 2.4 Strong density

In this section we want to introduce another property of quasi-metric spaces which is helpful to show that certain quasi-metric spaces are upper generated.

**Definition 2.4.1 (Strong density)** A subset $Y \subseteq X$ of a quasi-metric space $(X,d)$ is called *strongly dense*, if there is a constant $c \in \mathbb{N}$ such that for all $x \in X$, $y \in Y$ and $\varepsilon > 0$ there is a $z \in Y$ such that $d_\star(z,y) < c \cdot d(x,y) + \varepsilon$ and $d(x,z) < \varepsilon$.

If $Y$ is strongly dense in $(X,d)$, then it is dense in the lower topology, induced by $d$. If $(X,d)$ is a metric space, then $Y \subseteq X$ is strongly dense in $X$, if and only if it is dense in $X$. Thus, strong density is a generalization of the notion of density from metric to quasi-metric spaces. Intuitively, strong density guarantees that for each approximation $y \in Y$ of $x \in X$ we can find an arbitrarily good approximation $z \in Y$ of $x$ which is not too far away (in terms of the associated metric $d_\star$) from the previous approximation $y$. One can show that there are quasi-metric spaces which are not strongly dense in themselves (see Example 2.5.3). The following lemma will help us to prove the next proposition.

**Lemma 2.4.2** *Let $(X,d)$ be a quasi-metric space, let $x \in X$, let $Y \subseteq X$ and consider the set $U := \{y \in Y : x \sqsubseteq y\}$. If $V, W \subseteq U$ are such that $V$ is dense in $U$ with respect to the upper topology, induced by $d$, and $x = \inf W$, then $x = \inf V$.*

**Proof.** Since $V \subseteq U$, it follows that $x$ is a lower bound of $V$. We have to show that $x$ is a maximal lower bound of $V$. Let $y \in X$ be another lower bound of $V$, i.e. $y \sqsubseteq v$ and thus $d(y,v) = 0$ for all $v \in V$. Since $V$ is dense in $U$, there is some $v \in V$ for each $\varepsilon > 0$ and $w \in W \subseteq U$ such that $d(v,w) < \varepsilon$. We obtain

$$d(y,w) \leq d(y,v) + d(v,w) = d(v,w) < \varepsilon.$$

Hence $d(y,w) = 0$ and thus $y \sqsubseteq w$ for all $w \in W$. Consequently, $y \sqsubseteq \inf W = x$ and hence $x = \inf V$ follows. □

Now we can formulate and prove our topological condition which guarantees upper generation.

**Proposition 2.4.3 (Upper generation)** *If $(X,d)$ is a quasi-metric space with a strongly dense subset $Y \subseteq X$ such that $(Y, d_\star)$ is complete and separable and $x = \inf\{y \in Y : x \sqsubseteq y\}$ for each $x \in X$, then $(X,Y,d)$ is upper generated.*

**Proof.** Let $Y \subseteq X$ be strongly dense in $(X,d)$ with constant $c \geq 1$ and let $D \subseteq Y$ be a countable dense subset in $(Y, d_\star)$. We fix $x \in X$. By the previous lemma it suffices to prove that $U := \{y \in Y : x \sqsubseteq y\}$ has a countable subset $V$ which is dense in $U$ with respect to the upper topology, induced by $d$. For each $y_0 \in D$ and $l \in \mathbb{N}$ such that $d(x, y_0) < \frac{1}{c} \cdot 2^{-l-2}$ we choose a sequence $(y_n)_{n\in\mathbb{N}}$ in $Y$ with the same $y_0$ as first component such that

$$d_\star(y_{k+1}, y_k) < 2^{-k-l-1} \text{ and } d(x, y_k) < \frac{1}{c} \cdot 2^{-k-l-2} \quad (*)$$

for all $k \in \mathbb{N}$. By induction one can prove that such a sequence exists since $Y \subseteq X$ is strongly dense with constant $c$. Moreover, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and since $(Y, d_\star)$ is complete, the limit $y := \lim_{n\to\infty} y_n$ exists in $Y$. We obtain

$$d(x, y) \leq d(x, y_k) + d_\star(y_k, y) < \frac{1}{c} \cdot 2^{-k-l-2} + 2^{-k-l} < 2^{-k-l+1}$$

for all $k \in \mathbb{N}$ and hence $d(x, y) = 0$ and $x \sqsubseteq y$. Let $V$ be the set of all these limits $y$ for arbitrary but admissible starting value $y_0 \in D$ and $l \in \mathbb{N}$, as described above. Then $V \subseteq U$ and $V$ is countable since $D$ and $\mathbb{N}$ are. We will show that $V$ is dense in $U$ with respect to the metric topology induced by $d_\star$ and hence with respect to the upper topology. Therefore, let $y' \in U$ and $\varepsilon > 0$. Then there is some $l \in \mathbb{N}$ and $y_0 \in D$ such that $d_\star(y', y_0) < \frac{1}{c} \cdot 2^{-l-2}$ and $2^{-l+1} < \varepsilon$. Hence $d(x, y_0) < \frac{1}{c} \cdot 2^{-l-2}$ and there is a corresponding sequence $(y_n)_{n\in\mathbb{N}}$ which fulfills $(*)$ and $y := \lim_{n\to\infty} y_n \in V$. We obtain

$$d_\star(y', y) \leq d_\star(y', y_0) + d_\star(y_0, y) < \frac{1}{c} \cdot 2^{-l-2} + 2^{-l} < 2^{-l+1} < \varepsilon.$$

This completes the proof. □

## 2.5 The hyperspace of compact subsets

In this section we want to show that the set $\mathcal{K}(X)$ of non-empty compact subsets of a metric space $(X, d)$ can be considered as a generated quasi-metric space in a natural way. We will prove that this space is continuous from above and strongly dense in itself. Therefore, we consider the following quasi-metric, which consists of "one half" of the Hausdorff distance:

$$d'_\mathcal{K} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}, (A, B) \mapsto \sup_{a\in A} \inf_{b\in B} d(a, b).$$

In some references, $d'_\mathcal{K}$ is called the "excess of $A$ over $B$ with respect to $d$" (cf. [Bee93]). The metric, associated with $d'_\mathcal{K}$ is just the *Hausdorff metric* $d_\mathcal{K}$, i.e.

$$d_\mathcal{K}(A, B) = \max\{d'_\mathcal{K}(A, B), \overline{d'_\mathcal{K}}(A, B)\}.$$

The topology generated by $d_{\mathcal{K}}$ is the *Vietoris topology* on $\mathcal{K}(X)$ and thus we call the lower and upper topology generated by $d'_{\mathcal{K}}$ the *lower Vietoris topology* and the *upper Vietoris topology*, respectively. In the next lemma we will prove

$$d'_{\mathcal{K}}(A,B)=0 \iff A\subseteq B,$$

hence the partial order induced by $d'_{\mathcal{K}}$ is just the ordinary inclusion $\subseteq$. Here and in the following we will denote by

$$d_A : X \to \mathbb{R}, x \mapsto \inf_{a\in A} d(a,x)$$

the *distance function* of a subset $A \subseteq X$.

**Lemma 2.5.1** *$d'_{\mathcal{K}}$ is a quasi-metric on $\mathcal{K}(X)$ with induced partial order "$\subseteq$".*

**Proof.** First we prove that "$\subseteq$" is the partial order induced by $d'_{\mathcal{K}}$. Let $A,B\in\mathcal{K}(X)$ and $d'_{\mathcal{K}}(A,B)=0$. Now, $a\in A$ implies $d_B(a)=0$, i.e. $a\in B$ since $B$ is closed. Thus, $A\subseteq B$. If, on the other hand, $A\subseteq B$, then $d_B(a)=0$ for all $a\in A$ and thus $d'_{\mathcal{K}}(A,B)=0$. Now, we prove that $d'_{\mathcal{K}}$ is a quasi-metric.

(1) Obviously, $d'_{\mathcal{K}}(A,A)=0$ since $A\subseteq A$ for all $A\in\mathcal{K}(X)$.

(2) $d'_{\mathcal{K}}(A,B)=d'_{\mathcal{K}}(B,A)=0$ implies $A\subseteq B$ and $B\subseteq A$ and thus $A=B$.

(3) Let $A,B,C\in\mathcal{K}(X)$. Since $A,B$ are compact, there are $a'\in A$, $b'\in B$ such that $d_C(a')=\sup_{a\in A} d_C(a)$ and $d(a',b')=d_B(a')$. Now

$$\begin{aligned} d'_{\mathcal{K}}(A,C) &= d_C(a') \\ &\leq d(a',b')+d_C(b') \\ &= d_B(a')+d_C(b') \\ &\leq d'_{\mathcal{K}}(A,B)+d'_{\mathcal{K}}(B,C). \end{aligned}$$

□

For the corresponding infimum and supremum we obtain (in case of existence)

$$\sup_{n\in\mathbb{N}} A_n = \overline{\bigcup_{n=0}^{\infty} A_n} \quad\text{and}\quad \inf_{n\in\mathbb{N}} A_n = \bigcap_{n=0}^{\infty} A_n.$$

Here, by $\overline{A}$ we denote the *topological closure* of a set $A$. Obviously, $(\mathcal{K}(X),\subseteq)$ is a sup-semi-lattice but neither an inf-semi-lattice nor sup- or inf-complete in general. (It is easy to see that $(\mathcal{K}(X)\cup\{\emptyset\},\subseteq)$ is an inf-semi-lattice and inf-complete, but we will not use these facts.) In case that $X$ is compact $(\mathcal{K}(X),\subseteq)$ is also sup-complete. The following theorem yields the announced results on the quasi-metric space $(\mathcal{K}(X),d'_{\mathcal{K}})$ and its conjugate. We will sometimes write for short $\mathcal{K}:=\mathcal{K}(X)$.

**Theorem 2.5.2** *If $(X, d)$ is a separable metric space, then $(\mathcal{K}(X), \mathcal{K}(X), d'_{\mathcal{K}})$ is a quasi-metric space, which is upper generated and continuous from above, and $\mathcal{K}(X)$ is strongly dense in $(\mathcal{K}(X), d'_{\mathcal{K}})$. The same holds for the conjugate quasi-metric space $(\mathcal{K}(X), \mathcal{K}(X), \overline{d'_{\mathcal{K}}})$.*

**Proof.** Obviously, $(\mathcal{K}, \mathcal{K}, d'_{\mathcal{K}})$ is lower and upper generated.

We start to prove that $(\mathcal{K}, \mathcal{K}, d'_{\mathcal{K}})$ is consistent from below with respect to the lower topology. Let $(A_n)_{n\in\mathbb{N}}$ be an increasing chain in $\mathcal{K}$, i.e. $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, such that $A := \sup_{n\in\mathbb{N}} A_n = \overline{\bigcup_{n=0}^{\infty} A_n}$ is compact and let $\varepsilon > 0$. Then $\bigcup_{n=0}^{\infty} A_n$ is totally bounded and there is a finite subset $B \subseteq \bigcup_{n=0}^{\infty} A_n$ such that $d'_{\mathcal{K}}(A, B) < \varepsilon$. Since $B$ is finite and $(A_n)_{n\in\mathbb{N}}$ is increasing, there is some $k \in \mathbb{N}$ such that $B \subseteq A_n$ for all $n \geq k$. Thus

$$d'_{\mathcal{K}}(A, A_n) \leq d'_{\mathcal{K}}(A, B) + d'_{\mathcal{K}}(B, A_n) < \varepsilon$$

for all $n \geq k$, i.e. $(A_n)_{n\in\mathbb{N}}$ converges to $A$ with respect to the lower topology induced by $d'_{\mathcal{K}}$.

Now we prove that $(\mathcal{K}, \mathcal{K}, d'_{\mathcal{K}})$ is consistent from above with respect to the upper topology. Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing chain in $\mathcal{K}$, i.e. $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, such that $A := \inf_{n\in\mathbb{N}} A_n = \bigcap_{n=0}^{\infty} A_n \neq \emptyset$ and let $\varepsilon > 0$. Then $B := \{x \in X : d_A(x) < \frac{\varepsilon}{2}\}$ is open and $\{B\} \cup \{X \setminus A_n : n \in \mathbb{N}\}$ is an open covering of $A_0$. Since $A_0$ is compact, there is a finite subcovering, i.e. there is a $k \in \mathbb{N}$ such that $\{B\} \cup \{X \setminus A_n : n < k\}$ covers $A_0$. Since $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, we obtain $A_n \subseteq B$ for all $n \geq k$ and

$$d'_{\mathcal{K}}(A_n, A) = \sup_{a\in A_n} d_A(a) \leq \sup_{a\in B} d_A(a) \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $n \geq k$, i.e. $(A_n)_{n\in\mathbb{N}}$ converges to $A$ with respect to the upper topology induced by $d'_{\mathcal{K}}$.

By Proposition 2.3.4 it follows that $(\mathcal{K}, \mathcal{K}, d'_{\mathcal{K}})$ is continuous from above. By duality it follows that the conjugate space $(\mathcal{K}, \mathcal{K}, \overline{d'_{\mathcal{K}}})$ is upper generated and by Proposition 2.3.4 it follows that it is continuous from above.

Finally, we prove that $\mathcal{K}$ is strongly dense in $(\mathcal{K}, d'_{\mathcal{K}})$ and $(\mathcal{K}, \overline{d'_{\mathcal{K}}})$ with constant $c := 1$. Let $A, B \in \mathcal{K}$ and $\varepsilon > 0$. Then $C := A \cup B$ is a non-empty compact subset of $X$, $d'_{\mathcal{K}}(A, C) = 0 < \varepsilon$ and $d_{\mathcal{K}}(C, B) = d'_{\mathcal{K}}(C, B) = d'_{\mathcal{K}}(A, B) < c \cdot d'_{\mathcal{K}}(A, B) + \varepsilon$. On the other hand,

$$C' := \{x \in A : (\exists y \in B) d(x, y) \leq \overline{d'_{\mathcal{K}}}(A, B)\}$$

is a non-empty compact subset of $X$. Since $C' \subseteq A$ we obtain $\overline{d'_{\mathcal{K}}}(A, C') = 0 < \varepsilon$ and by definition of $C'$ we obtain $d_{\mathcal{K}}(C', B) \leq \overline{d'_{\mathcal{K}}}(A, B) < c \cdot \overline{d'_{\mathcal{K}}}(A, B) + \varepsilon$. □

This space is an example of a space where it does not suffice to consider a countable dense subset as generating set in general. If $X$ is non-countable, then $(\mathcal{K}(X), G, d'_{\mathcal{K}})$ is not lower generated for any countable subset $G \subseteq \mathcal{K}(X)$, since $\{x\} \in \mathcal{K}(X)$ for all $x \in X$ and not all such singleton sets can be represented as $\{x\} = \overline{\bigcup_{n=0}^{\infty} A_n} = \sup_{n\in\mathbb{N}} A_n$ with subsets $A_n \in G$.

The final example of this section shows that the restriction of a space which is strongly dense in itself is not necessarily strongly dense in itself. By the previous theorem, the set $\mathcal{K}(\mathbb{N})$ (where $\mathbb{N}$ is the metric space of the natural numbers endowed with the Euclidean metric) is dense in $(\mathcal{K}(\mathbb{N}), \overline{d'_{\mathcal{K}}})$. In the following example we consider a subspace of this space.

**Example 2.5.3** *Let* $X := \{A_n : n \in \mathbb{N}\}$ *with the sets* $A_0 := \{0\}$ *and* $A_n := \{n, n+1, n+2, ..., n^2+1\}$ *for all* $n \geq 1$ *and let* $d : X \times X \to \mathbb{R}$ *be given by* $d(A, B) := \sup_{b\in B} \inf_{a\in A} |a - b|$. *We claim that* $X$ *is not strongly dense in* $(X, d)$. *Let us assume that* $X$ *would be strongly dense with constant* $c \geq 1$ *and let* $A := A_c$, $B := A_0$ *and* $\varepsilon := \frac{1}{2}$. *Then there exists some* $C \in X$ *such that*

$$d(A, C) < \varepsilon \text{ and } d_\star(C, B) < c \cdot d(A, B) + \varepsilon.$$

*The only set* $C \in X$ *such that* $d(A, C) < \varepsilon$ *is* $C = A$ *since* $c \geq 1$. *But* $d_\star(C, B) = d(B, A) = c^2 + 1 > c^2 + \frac{1}{2} = c \cdot d(A, B) + \varepsilon$. *Contradiction!*

## 2.6 The spaces of semi-continuous functions

In this section we want to endow the sets $\mathcal{LSC}(X), \mathcal{USC}(X)$ of lower and upper semi-continuous functions $f : X \to \mathbb{R}$, defined on a metric space $X$, with a quasi-metric. Here, $f$ is called *lower semi-continuous*, if $f^{-1}(x, \infty)$ is open in $X$ and *upper semi-continuous*, if $f^{-1}(-\infty, x)$ is open in $X$ for all $x \in \mathbb{R}$. In other words, lower semi-continuous functions are just the continuous functions $f : X \to \mathbb{R}_<$ and upper semi-continuous functions are just the continuous functions $f : X \to \mathbb{R}_>$, where $\mathbb{R}_<$ is equipped with the lower and $\mathbb{R}_>$ with the upper Euclidean topology. Obviously, a function $f : X \to \mathbb{R}$ is continuous, if and only if it is lower and upper semi-continuous, i.e.

$$\mathcal{C}(X) = \mathcal{LSC}(X) \cap \mathcal{USC}(X).$$

We prove that a separable metric space is locally compact, if and only if it admits an *exhausting sequence* $(K_i)_{i\in\mathbb{N}}$, i.e. if and only if it can be represented as a countable union $X = \bigcup_{i=0}^{\infty} K_i$ of compact subsets $K_i \subseteq X$ such that $K_i \subseteq K_{i+1}^\circ$ for all $i \in \mathbb{N}$. Here $A^\circ$ denotes the *interior* of $A$. We recall that a Hausdorff space is called *locally compact*, if each point admits a compact neighbourhood.

**Lemma 2.6.1** *Let $(X, d)$ be a separable metric space. The following conditions are equivalent:*

*(1) $X$ is locally compact,*

*(2) $X$ admits an exhausting sequence $(K_n)_{n\in\mathbb{N}}$.*

**Proof.** "(1)$\Longrightarrow$(2)" Let $X$ be locally compact and separable. Then $X$ admits a base of relatively compact open subsets and since $X$ is second countable, there is a countable subset of this base which is a base too (cf. Theorem 1.1.15 [Eng89]). Thus, $X = \bigcup_{i=0}^{\infty} K_i'$ is a countable union of compact subsets $K_i' \subseteq X$. For any compact subset $K' \subseteq X$ there exists an open subset $U \subseteq X$ and a compact subset $K \subseteq X$ such that $K' \subseteq U \subseteq K$ (cf. Theorem 3.3.2 in [Eng89]). Using this fact we choose $K_0 := K_0'$, some open set $U_{n+1} \subseteq X$ and a compact set $K_{n+1} \subseteq X$ such that $K_{n+1}' \cup K_n \subseteq U_{n+1} \subseteq K_{n+1}$ for all $n \in \mathbb{N}$. We obtain $X = \bigcup_{i=0}^{\infty} K_i$ and $K_i \subseteq K_{i+1}^{\circ}$ for all $i \in \mathbb{N}$. Thus, $(K_n)_{n\in\mathbb{N}}$ is an exhausting sequence of $X$.

"(2)$\Longrightarrow$(1)" If $X$ admits an exhausting sequence $(K_n)_{n\in\mathbb{N}}$, then for any point $x \in X = \bigcup_{i=0}^{\infty} K_i$ there is some $n \in \mathbb{N}$ such that $x \in K_n \subseteq K_{n+1}^{\circ}$. Thus, $K_{n+1}$ is a compact neighbourhood of $x$ and thus $X$ is locally compact. □

In case that $X$ is a locally compact separable metric space with exhausting sequence $(K_i)_{i\in\mathbb{N}}$, we define

$$d'(f, g) := \sum_{i=0}^{\infty} 2^{-i-1} \left| \frac{f \dot{-} g}{1 + (f \dot{-} g)} \right|_{K_i}$$

for all functions $f, g : X \to \mathbb{R}$. Here $f \dot{-} g : X \to \mathbb{R}$ is defined by

$$(f \dot{-} g)(x) := f(x) \dot{-} g(x) = \max\{0, f(x) - g(x)\}$$

for all $x \in X$ and $|f|_K := \sup_{x\in K} |f(x)|$. Then

$$d_{\mathcal{USC}} : \mathcal{USC}(X) \times \mathcal{USC}(X) \to \mathbb{R}, (f, g) \mapsto d'(f, g)$$

and $d_{\mathcal{LSC}} : \mathcal{LSC}(X) \times \mathcal{LSC}(X) \to \mathbb{R}, (f, g) \mapsto d'(g, f) = d'(-f, -g)$ are quasi-metrics, as we will prove below. We will only discuss the upper semi-continuous functions in the following. According to the fact that $f \in \mathcal{LSC}(X) \iff -f \in \mathcal{USC}(X)$, the lower case can be treated analogously. We will prove

$$d_{\mathcal{USC}}(f, g) = 0 \iff (\forall x \in X)(f(x) \leq g(x)) \iff f \leq g,$$

and hence the partial order induced by $d_{\mathcal{USC}}$ is just the ordinary pointwise partial order $\leq$ for functions.

**Lemma 2.6.2** *$d_{\mathcal{USC}}$ is a quasi-metric on $\mathcal{USC}(X)$ and "$\leq$" is the associated partial order.*

**Proof.** First we proof $d_{\mathcal{USC}}(f,g) = 0 \iff f \leq g$. Let $d_{\mathcal{USC}}(f,g) = 0$ and $x \in X$. Then there is some $i \in \mathbb{N}$ such that $x \in K_i$. Now $d_{\mathcal{USC}}(f,g) = 0$ implies $\left|\frac{f \dot{-} g}{1+(f \dot{-} g)}\right|_{K_i} = 0$ and thus especially $\frac{f(x) \dot{-} g(x)}{1+(f(x) \dot{-} g(x))} = 0$. Hence $f(x) \leq g(x)$ and altogether $f \leq g$. On the other hand, $f \leq g$ implies $f(x) \leq g(x)$ for all $x \in X$ and thus $f \dot{-} g = 0$ and $d_{\mathcal{USC}}(f,g) = 0$. Now we show that $d_{\mathcal{USC}}$ is a quasi-metric.

(1) Obviously, $d_{\mathcal{USC}}(f,f) = 0$ for all $f \in \mathcal{USC}(X)$, since $f \leq f$.

(2) $d_{\mathcal{USC}}(f,g) = d_{\mathcal{USC}}(g,f) = 0$ implies $f \leq g$ and $g \leq f$ and thus $f = g$.

(3) Let $f, g, h \in \mathcal{USC}(X)$. For any $x \in X$ we have

$$f(x) \dot{-} h(x) \leq (f(x) \dot{-} g(x)) + (g(x) \dot{-} h(x))$$

since $(x,y) \mapsto x \dot{-} y$ is a quasi-metric on $\mathbb{R}$. Moreover, the function $F :\subseteq \mathbb{R} \to \mathbb{R}, x \mapsto \frac{x}{1+x}$ is strictly increasing on $[0,\infty)$ and we obtain $F(x+y) \leq F(x) + F(y)$ for all $x, y \in [0,\infty)$, since

$$\frac{x}{1+x} + \frac{y}{1+y} \geq \frac{x}{1+x+y} + \frac{y}{1+x+y} = \frac{x+y}{1+x+y}.$$

This implies

$$\begin{aligned} F(f(x) \dot{-} h(x)) &\leq F((f(x) \dot{-} g(x)) + (g(x) \dot{-} h(x))) \\ &\leq F(f(x) \dot{-} g(x)) + F(g(x) \dot{-} h(x)) \end{aligned}$$

for all $x \in X$ and thus

$$\left|\frac{f \dot{-} h}{1+(f \dot{-} h)}\right|_{K_i} \leq \left|\frac{f \dot{-} g}{1+(f \dot{-} g)}\right|_{K_i} + \left|\frac{g \dot{-} h}{1+(g \dot{-} h)}\right|_{K_i}$$

for all $i \in \mathbb{N}$ and hence

$$d_{\mathcal{USC}}(f,h) \leq d_{\mathcal{USC}}(f,g) + d_{\mathcal{USC}}(g,h).$$

□

Since "$\leq$" is the partial order induced by $d_{\mathcal{USC}}$, we obtain

$$\mathrm{Inf}(f_n)_{n\in\mathbb{N}}(x) = \inf_{n\in\mathbb{N}} f_n(x)$$

for all sequences $(f_n)_{n\in\mathbb{N}} \in \mathcal{USC}(X)^{\mathbb{N}}$ and $x \in X$, provided the right-hand values exist for all $x \in X$. The partial ordered space $(\mathcal{USC}(X), \leq)$ is not complete, but it is a lattice, which will be helpful in the following.

One can prove that $\sup_{x\in K} f(x)$ exists for each $f \in \mathcal{USC}(X)$ and each compact set $K \subseteq X$ (cf. Chapter IV §6 of [Bou66a] for the theory of semi-continuous functions and, in particular, Theorem 3 for the previous statement). Furthermore, $f \dot{-} g = \max(0, f-g) \in \mathcal{USC}(X)$ for all $f \in \mathcal{USC}(X)$, $g \in \mathcal{C}(X)$. Thus we obtain

$$d_{\mathcal{USC}}|_{\mathcal{USC}(X)\times\mathcal{C}(X)}(f,g) = \sum_{i=0}^{\infty} 2^{-i-1} \frac{|f \dot{-} g|_{K_i}}{1 + |f \dot{-} g|_{K_i}}.$$

In Theorem 2.6.4 we will see that $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is an upper generated quasi-metric space. In the next step we will show that the weak upper topology of the space $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is a well-known topology: the topology on $\mathcal{USC}(X)$, with the sets

$$B(K,q) := \{f \in \mathcal{USC}(X) : f(K) \subseteq (-\infty, q)\}$$

for compact $K \subseteq X$ and $q \in \mathbb{R}$ as subbase, is called the *upper compact open topology*. The upper compact open topology is the compact open topology with respect to $X$ and $\mathbb{R}_>$.

**Proposition 2.6.3** *The weak upper topology of* $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ *is the upper compact open topology.*

**Proof.** Let $(K_i)_{i\in\mathbb{N}}$ be an exhausting sequence of $X$, let $\tau$ denote the upper compact open topology and let $\tau_w$ denote the weak upper topology of $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$. We have to prove $\tau = \tau_w$.

"$\subseteq$" Let $K \subseteq X$ be compact and $q \in \mathbb{R}$ and $f \in B(K,q)$, i.e. $f(K) \subseteq (-\infty, q)$. Since there exists some $y \in K$ such that $f(y) = \sup_{x\in K} f(x) =: s$, we obtain $s < q$. Then $\delta := q - s > 0$ and since $K$ is compact, there exists some $j \in \mathbb{N}$ with $K \subseteq K_j^\circ$. Let $t_i := \sup_{x\in K_i} f(x)$ for all $i \in \mathbb{N}$ and define $l : X \to \mathbb{R}$ by

$$\begin{cases} l|_K(x) & := & s \\ l|_{K_j\setminus K}(x) & := & t_j \\ l|_{K_{j+i+1}\setminus K_{j+i}}(x) & := & t_{j+i+1} \end{cases}$$

for all $i \in \mathbb{N}$ and $x \in X$. Let $r \in \mathbb{R}$, $L := \min\{i : t_i > r\}$. We obtain

$$l^{-1}(r,\infty) = \begin{cases} X & \text{if } s > r \\ X \setminus K & \text{if } t_j > r \text{ and } s \leq r \\ X \setminus K_{L-1} & \text{if } L > j \\ \emptyset & \text{if } (\forall i)\ t_i \leq r \end{cases}.$$

Thus $l$ is lower semi-continuous and since $f \leq l$ by a Theorem of Hahn, Tong and others (cf. Problem 1.7.15 (b) in [Eng89]) there exists some continuous function $g \in \mathcal{C}(X)$ such that $f \leq g \leq l$. Let $\varepsilon < 2^{-j-1}\frac{\frac{\delta}{2}}{1+\frac{\delta}{2}}$. Since $f \leq g$ we obtain $d_{\mathcal{USC}}(f,g) = 0$ and $f \in B_{>}(g,\varepsilon)$. Now let $h \in B_{>}(g,\varepsilon)$. Then $d_{\mathcal{USC}}(h,g) < \varepsilon$ and especially

$$2^{-j-1}\frac{|h \dot{-} g|_{K_j}}{1+|h \dot{-} g|_{K_j}} < \varepsilon < 2^{-j-1}\frac{\frac{\delta}{2}}{1+\frac{\delta}{2}}.$$

Thus, $|h \dot{-} g|_{K_j} < \frac{\delta}{2}$ and especially $\sup h(K) \leq s + \frac{\delta}{2} = s + \frac{q-s}{2} < q$. Hence, $h(K) \subseteq (-\infty, q)$ and $h \in B(K,q)$. Altogether, $f \in B_{>}(g,\varepsilon) \subseteq B(K,q)$ and thus $\tau \subseteq \tau_w$.

"$\supseteq$" Let $g \in \mathcal{C}(X)$ and $\varepsilon > 0$ and let $f \in B_{>}(g,\varepsilon)$. Then there is some $j \in \mathbb{N}$ and some $\delta > 0$ such that $2^{-j-1} + \frac{\delta}{1+\delta} + d_{\mathcal{USC}}(f,g) < \varepsilon$. Since $g$ is continuous, there is an open neighbourhood $U_x$ of each point $x \in K_j$ such that $\operatorname{diam} g(U_x) < \frac{\delta}{3}$. Since $K_j$ is compact, the covering $\{U_x : x \in K_j\}$ of $K_j$ admits a finite subcovering, i.e. there are points $x_1, ..., x_n \in K_j$ such that $K_j \subseteq \bigcup_{i=1}^{n} U_{x_i}$. Now let

$$L_{i\iota} := \overline{U_{x_i}} \cap K_\iota \quad \text{and} \quad q_{i\iota} := \sup g(L_{i\iota}) + |f \dot{-} g|_{K_\iota} + \frac{\delta}{2}$$

for all $i = 1, ..., n$ and $\iota = 0, ..., j$. Then all sets $L_{i\iota}$ are compact and $K_\iota = \bigcup_{i=1}^{n} L_{i\iota}$ for all $\iota = 0, ..., j$. For all $i = 1, ..., n$ and $\iota = 0, ..., j$ such that $L_{i\iota} \neq \emptyset$, the number $q_{i\iota}$ exists and $f \in B(L_{i\iota}, q_{i\iota})$ by definition of $L_{i\iota}$ and $q_{i\iota}$. Moreover, if $h \in B(L_{i\iota}, q_{i\iota})$ for all such $i, \iota$, we obtain

$$\begin{aligned} |h \dot{-} g|_{L_{i\iota}} &\leq q_{i\iota} - \inf g(L_{i\iota}) \\ &\leq q_{i\iota} - (\sup g(L_{i\iota}) - \operatorname{diam} g(L_{i\iota})) \\ &< q_{i\iota} - (q_{i\iota} - |f \dot{-} g|_{K_\iota} - \delta) \\ &= |f \dot{-} g|_{K_\iota} + \delta \end{aligned}$$

and thus $|h \dot{-} g|_{K_\iota} < |f \dot{-} g|_{K_\iota} + \delta$ for all $\iota = 0, ..., j$. It follows

$$\begin{aligned} d_{\mathcal{USC}}(h,g) &= \sum_{i=0}^{\infty} 2^{-i-1}\frac{|h \dot{-} g|_{K_i}}{1+|h \dot{-} g|_{K_i}} \\ &\leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1}\frac{|h \dot{-} g|_{K_i}}{1+|h \dot{-} g|_{K_i}} \\ &< 2^{-j-1} + \frac{\delta}{1+\delta} + \sum_{i=0}^{j} 2^{-i-1}\frac{|f \dot{-} g|_{K_i}}{1+|f \dot{-} g|_{K_i}} \end{aligned}$$

$$
\begin{aligned}
&< \; 2^{-j-1} + \frac{\delta}{1+\delta} + d_{\mathcal{USC}}(f,g) \\
&< \; \varepsilon,
\end{aligned}
$$

i.e. $h \in B_>(g,\varepsilon)$. Altogether, we have proved

$$f \in \bigcap \{B(L_{i\iota}, q_{i\iota}) : i = 1,...,n, \iota = 0,...,j, L_{i\iota} \neq \emptyset\} \subseteq B_>(g,\varepsilon)$$

and thus $\tau_w \subseteq \tau$. □

By duality it follows that the weak upper topology of $(\mathcal{LSC}(X), \mathcal{C}(X), d_{\mathcal{LSC}})$ is the lower compact open topology (i.e. the compact open topology with respect to $X$ and $\mathbb{R}_<$). The metric $(d'_{\mathcal{C}})_\star$, associated with $d'_{\mathcal{C}} := d_{\mathcal{USC}}|_{\mathcal{C}(X)\times\mathcal{C}(X)}$, is equivalent to the metric $d_{\mathcal{C}}$, defined by

$$d_{\mathcal{C}} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}, (f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{|f-g|_{K_i}}{1+|f-g|_{K_i}}.$$

The equivalence follows since $2\max\{x,y\} \geq x+y$ and thus

$$(d'_{\mathcal{C}})_\star \leq d_{\mathcal{C}} \leq 2(d'_{\mathcal{C}})_\star. \tag{2.1}$$

It is easy to see that $(\mathcal{C}(X), d_{\mathcal{C}})$ is a complete metric space. The following result provides some essential topological properties of the space of semi-continuous functions.

**Theorem 2.6.4** *Let $(X,d)$ be a locally compact separable metric space. The quasi-metric space $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is upper generated, continuous from above and $\mathcal{C}(X)$ is strongly dense in this space. The same properties hold in case of $\mathcal{LSC}$ instead of $\mathcal{USC}$.*

**Proof.** Let $(K_i)_{i\in\mathbb{N}}$ be the exhausting sequence of $X$.

First, we prove that $\mathcal{C}(X)$ is strongly dense in $(\mathcal{USC}(X), d_{\mathcal{USC}})$ with constant $c := 4$. Let $f \in \mathcal{USC}(X)$, $g \in \mathcal{C}(X)$ and $\varepsilon > 0$. Let $\delta_i := |f \dot{-} g|_{K_i}$ for all $i \in \mathbb{N}$. Then $\delta_i \leq \delta_{i+1}$ for all $i$. We define functions $l, u, u' : X \to \mathbb{R}$ by

$$\begin{cases} l|_{K_0}(x) & := \; \delta_1 \\ l|_{K_{i+1}\setminus K_i}(x) & := \; \delta_{i+2} \end{cases}$$

$$\begin{cases} u|_{K_0^\circ}(x) & := \; \delta_0 \\ u|_{K_{i+1}^\circ\setminus K_i^\circ}(x) & := \; \delta_{i+1} \end{cases}$$

$$\begin{cases} u'|_{K_0^\circ}(x) & := \; \delta_1 \\ u'|_{K_{i+1}^\circ\setminus K_i^\circ}(x) & := \; \delta_{i+2} \end{cases}$$

for all $i \in \mathbb{N}$ and $x \in X$. Let $r \in \mathbb{R}$,

$$L := \min\{i : \delta_i > r\} \text{ and } U := \max\{i : \delta_i < r\}.$$

We obtain

$$l^{-1}(r,\infty) = \begin{cases} X & \text{if} \quad \delta_1 > r \\ X \setminus K_{L-2} & \text{if} \quad L \geq 2 \\ \emptyset & \text{if} \quad (\forall i)\ \delta_i \leq r \end{cases} \quad \text{and}$$

$$u^{-1}(-\infty,r) = \begin{cases} \emptyset & \text{if} \quad \delta_0 \geq r \\ K_U^\circ & \text{if} \quad U \geq 0 \\ X & \text{if} \quad (\forall i)\ \delta_i < r \end{cases} .$$

Thus $l \in \mathcal{LSC}(X)$, $u \in \mathcal{USC}(X)$. Analogously, one can prove $u' \in \mathcal{USC}(X)$. We obtain $u \leq l \leq u'$ since $\delta_i \leq \delta_{i+1}$ for all $i \in \mathbb{N}$. Furthermore, $f \leq g+u$. By a Theorem of Hahn, Tong and others (cf. Problem 1.7.15(b) in [Eng89]) there is a function $h \in \mathcal{C}(X)$ such that

$$g + u \leq h \leq g + l.$$

We obtain $f \leq h$ and thus $d_{\mathcal{USC}}(f,h) = 0 < \varepsilon$. Moreover, there is some $j \in \mathbb{N}$ such that $2^{-j-1} < \varepsilon$. In particular, $g \leq h \leq g + u'$ and thus

$$\begin{aligned}
(d'_{\mathcal{C}})_\star(h,g) &\leq d'_{\mathcal{C}}(h,g) \\
&\leq d_{\mathcal{USC}}(g+u',g) \\
&\leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} \frac{|(g+\delta_{i+2}) \dot{-} g|_{K_i}}{1 + |(g+\delta_{i+2}) \dot{-} g|_{K_i}} \\
&\leq 2^{-j-1} + 4 \cdot \sum_{i=0}^{j} 2^{-i-3} \frac{\delta_{i+2}}{1+\delta_{i+2}} \\
&\leq 2^{-j-1} + 4 \cdot \sum_{i=0}^{j+2} 2^{-i-1} \frac{|f \dot{-} g|_{K_i}}{1 + |f \dot{-} g|_{K_i}} \\
&< \varepsilon + c \cdot d_{\mathcal{USC}}(f,g).
\end{aligned}$$

Thus, $\mathcal{C}(X)$ is strongly dense in $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$.

In the previous step we have especially seen that for each $f \in \mathcal{USC}(X)$ there is a function $h \in \mathcal{C}(X)$ such that $f \leq h$. By a Theorem of Bourbaki (cf. Proposition 5 in Chapter IX §1.6 of [Bou66b]) it follows that

$$f = \inf\{g \in \mathcal{C}(X) : f \leq g\}.$$

Since $(\mathcal{C}(X), d_{\mathcal{C}})$ is complete and also separable, we can conclude by Proposition 2.4.3 that $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is upper generated.

Finally, we prove that $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is continuous from above. Let $(g_n)_{n\in\mathbb{N}} \in \mathcal{C}(X)^{\mathbb{N}}$ be a decreasing chain, i.e. $g_{n+1} \leq g_n$ for all $n$, such that $f := \inf_{n\in\mathbb{N}} g_n$ exists. Then $f \in \mathcal{USC}(X)$ (cf. Theorem 4 in Chapter IV §6.2 of [Bou66a]). Let $h \in \mathcal{C}(X)$. We have to show

$$d_{\mathcal{USC}}(f, h) = \inf_{n\in\mathbb{N}} d_{\mathcal{USC}}(g_n, h).$$

Therefore, let $y := d_{\mathcal{USC}}(f, h)$ and $y_n := d_{\mathcal{USC}}(g_n, h)$ for all $n \in \mathbb{N}$. Since $d_{\mathcal{USC}}$ is isotone, $y \leq y_n$ follows for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$. We will show that there is some $n \in \mathbb{N}$ with $y_n \leq y + \varepsilon$. Let $j \in \mathbb{N}$ be such that $2^{-j-1} < \varepsilon/2$, let $\delta > 0$ be such that $\frac{\delta}{1+\delta} < \frac{\varepsilon}{2}$ and consider an arbitrary $i \in \mathbb{N}$. Since $s_n := g_n \dot{-} h$ is continuous for each $n \in \mathbb{N}$, there is a $x'_n \in K_i$ for each $n \in \mathbb{N}$ such that

$$s_n(x'_n) = |s_n|_{K_i}.$$

Since $K_i$ is compact there is a convergent subsequence $(x_n)_{n\in\mathbb{N}}$ of $(x'_n)_{n\in\mathbb{N}}$. Let $x := \lim_{n\to\infty} x_n$ and $t := f \dot{-} h$. Then $x \in K_i$ and $t = \inf_{n\in\mathbb{N}} s_n$, $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$ and there is some $k \in \mathbb{N}$ such that

$$s_k(x) - t(x) < \delta.$$

By continuity of $s_k$ there is some $m \geq k$ such that

$$s_k(x_n) - t(x) < \delta$$

for all $n \geq m$. Let $n_i \geq m$ be such that $x_m = x'_{n_i}$. Then $n_i \geq m \geq k$ and $s_k \geq s_{n_i}$. We obtain

$$\begin{aligned} |s_{n_i}|_{K_i} - |t|_{K_i} &= s_{n_i}(x'_{n_i}) - |t|_{K_i} \\ &\leq s_{n_i}(x_m) - t(x) \\ &\leq s_k(x_m) - t(x) \\ &< \delta. \end{aligned}$$

Let $n := \max\{n_0, ..., n_j\}$. Then $g_n \leq g_{n_i}$ for $i = 0, ..., j$ and

$$\begin{aligned} y_n = d_{\mathcal{USC}}(g_n, h) &\leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} \frac{|g_{n_i} \dot{-} h|_{K_i}}{1 + |g_{n_i} \dot{-} h|_{K_i}} \\ &= 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} \frac{|s_{n_i}|_{K_i}}{1 + |s_{n_i}|_{K_i}} \end{aligned}$$

$$
\begin{aligned}
&\leq\ 2^{-j-1}+\sum_{i=0}^{j}2^{-i-1}\left(\frac{|t|_{K_i}}{1+|t|_{K_i}}+\frac{\delta}{1+\delta}\right)\\
&\leq\ 2^{-j-1}+\frac{\delta}{1+\delta}+d_{\mathcal{USC}}(f,h)\\
&<\ \varepsilon+d_{\mathcal{USC}}(f,h)\\
&=\ \varepsilon+y,
\end{aligned}
$$

i.e. $y_n \leq y+\varepsilon$. This completes the proof. □

It is easy to prove that $(\mathcal{USC}(X),\mathcal{C}(X),d_{\mathcal{USC}})$ is not lower generated in general. Finally, we mention that the weak upper topology and the upper topology of $(\mathcal{USC}(X),\mathcal{C}(X),d_{\mathcal{USC}})$ do not coincide in general. In case of $X=[0,1]$ the space is continuous from above by the previous theorem and hence consistent from above with respect to the weak upper topology. The following example shows that the space is not consistent from above with respect to the upper topology and hence both topologies cannot coincide.

**Example 2.6.5** *Let $X=[0,1]$ with the constant exhausting sequence given by $K_i := X$. Let $f:[0,1]\to\mathbb{R}$ be defined by*

$$f(x) := \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ 1 & \text{else} \end{cases} .$$

*Then $f\in\mathcal{USC}(X)$ and $f\leq g$ implies $f(\frac{1}{2})=1\leq g(\frac{1}{2})$ for any $g\in\mathcal{C}(X)$ and thus $|g\dot{-}f|_{[0,1]}\geq 1$ and hence $d_{\mathcal{USC}}(g,f)\geq\frac{1}{2}$. Thus, there is no sequence $(g_n)_{n\in\mathbb{N}}$ with $f\leq g_n$ in $\mathcal{C}(X)$ which converges to $f$ with respect to the upper topology although $f=\inf\{g\in\mathcal{C}(X): f\leq g\}$. Altogether, this proves that $(\mathcal{USC}(X),\mathcal{C}(X),d_{\mathcal{USC}})$ is not consistent from above with respect to the upper topology.*

## 2.7 The hyperspace of closed subsets

In this section we want to endow the set $\mathcal{A}(X)$ of non-empty closed subsets of a locally compact separable metric space $(X,d)$ with a quasi-metric structure. Without loss of generality, we can restrict the investigation to nice metrics, which are, roughly speaking, metrics that are subordinated to the exhausting sequence $(K_i)_{i\in\mathbb{N}}$ of the space. For the concept of nice metrics cf. [Bee93].

**Definition 2.7.1 (Nice metric spaces)** A locally compact separable metric space $(X,d)$ with exhausting sequence $(K_i)_{i\in\mathbb{N}}$ is called *nice* with respect to $(K_i)_{i\in\mathbb{N}}$, if $d$ is bounded by 1 and if $x\in K_i$, $d(x,y)<1$ implies $y\in K_{i+1}^{\circ}$ for all $x,y\in X$ and $i\in\mathbb{N}$.

In other words $\bigcup_{x\in K_i} B(x,1) \subseteq K_{i+1}^\circ$ holds for nice metric spaces. If $(X,d)$ is a nice metric space, then we will sometimes say for short that $d$ is nice. In case that $(X,d)$ is a nice locally compact separable metric space with exhausting sequence $(K_i)_{i\in\mathbb{N}}$, we consider the following quasi-metric:

$$d'_{\mathcal{A}} : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}, (A,B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} |d_B \dot{-} d_A|_{K_i}.$$

Here $d_A : X \to \mathbb{R}, x \mapsto \inf_{a\in A} d(a,x)$ denotes the *distance function* of a subset $A \subseteq X$. We note that $d'_{\mathcal{A}}$ is well-defined since nice metrics are bounded by one. As in case of the compact subsets we obtain

$$d'_{\mathcal{A}}(A,B) = 0 \iff A \subseteq B,$$

hence the partial order induced by $d'_{\mathcal{A}}$ is just the ordinary inclusion $\subseteq$. We first prove that $d'_{\mathcal{A}}$ is actually a quasi-metric and "$\subseteq$" is its induced partial order.

**Lemma 2.7.2** *$d'_{\mathcal{A}}$ is a quasi-metric on $\mathcal{A}(X)$ with induced partial order "$\subseteq$".*

**Proof.** First we prove that $d'_{\mathcal{A}}(A,B) = 0 \iff A \subseteq B$. Let $d'_{\mathcal{A}}(A,B) = 0$ and $x \in A$. Then there is some $i \in \mathbb{N}$ such that $x \in K_i$ and $d'_{\mathcal{A}}(A,B) = 0$ implies $|d_B \dot{-} d_A|_{K_i} = 0$ and thus $0 = d_A(x) \geq d_B(x)$, i.e. $x \in B$. On the other hand, let $A \subseteq B$. Then $d_B(x) \leq d_A(x)$ for all $x \in X$ and thus $|d_B \dot{-} d_A|_{K_i} = 0$ for all $i \in \mathbb{N}$ and hence $d'_{\mathcal{A}}(A,B) = 0$. Now we show that $d'_{\mathcal{A}}$ is a quasi-metric.

(1) Obviously, $d'_{\mathcal{A}}(A,A) = 0$ since $A \subseteq A$.

(2) $d'_{\mathcal{A}}(A,B) = d'_{\mathcal{A}}(B,A) = 0$ implies $A \subseteq B$ and $B \subseteq A$ and thus $A = B$.

(3) Now let $A, B, C \in \mathcal{A}(X)$. For any $x \in X$ we obtain

$$d_C(x) \dot{-} d_A(x) \leq (d_B(x) \dot{-} d_A(x)) + (d_C(x) \dot{-} d_B(x))$$

and thus

$$|d_C \dot{-} d_A|_{K_i} \leq |d_B \dot{-} d_A|_{K_i} + |d_C \dot{-} d_B|_{K_i}$$

for all $i \in \mathbb{N}$ and hence $d'_{\mathcal{A}}(A,C) \leq d'_{\mathcal{A}}(A,B) + d'_{\mathcal{A}}(B,C)$.

□

For the corresponding infimum and supremum we obtain (in case of existence)

$$\sup_{n\in\mathbb{N}} A_n = \overline{\bigcup_{n=0}^{\infty} A_n} \text{ and } \inf_{n\in\mathbb{N}} A_n = \bigcap_{n=0}^{\infty} A_n.$$

Obviously, $(\mathcal{A}(X), \subseteq)$ is a sup-semi-lattice and sup-complete, but neither an inf-semi-lattice nor inf-complete in general. (It is easy to see that $(\mathcal{A}(X) \cup \{\emptyset\}, \subseteq)$ is an inf-semi-lattice and inf-complete, but we will not use these facts.) The following lemma provides a useful property of nice metrics.

**Lemma 2.7.3** *If $(X, d)$ is a nice locally compact separable metric space with exhausting sequence $(K_i)_{i\in\mathbb{N}}$, then*

$$|d_B \dot{-} d_A|_{K_i} = |d_B \dot{-} d_{A\cap K_{i+1}}|_{K_i} \leq \sup_{x\in A\cap K_{i+1}} d_B(x)$$

*for all $A, B \in \mathcal{A}(X)$ and $i \in \mathbb{N}$. (Here we tacitly assume that $d_\emptyset = 1$ and that the supremum on the right hand side is equal to $0$ if $A \cap K_{i+1} = \emptyset$.)*

**Proof.** We recall $|d_B \dot{-} d_A|_{K_i} = \sup_{x\in K_i} |d_B(x) \dot{-} d_A(x)|$. If there is some $x \in K_i$ such that $d_A(x) < 1$, then for any $\varepsilon > 0$ with $d_A(x) + \varepsilon < 1$ there is some $a \in A$ such that $d(a, x) < d_A(x) + \varepsilon$ and thus $a \in K_{i+1}$ since $d$ is nice. Thus, $|d_B \dot{-} d_A|_{K_i} = |d_B \dot{-} d_{A\cap K_{i+1}}|_{K_i}$. If $A \cap K_{i+1} = \emptyset$,then $|d_B \dot{-} d_{A\cap K_{i+1}}| = 0 = \sup_{x\in A\cap K_{i+1}} d_B(x)$. If $A \cap K_{i+1} \neq \emptyset$, then there is some $y \in K_i$ such that

$$d_B(y) \dot{-} d_{A\cap K_{i+1}}(y) = \sup_{x\in K_i} |d_B(x) \dot{-} d_{A\cap K_{i+1}}(x)|$$

and there is some $a \in A \cap K_{i+1}$ such that $d(a, y) = d_{A\cap K_{i+1}}(y)$. We obtain

$$\begin{aligned}
\sup_{x\in A\cap K_{i+1}} d_B(x) &\geq d_B(a) \\
&\geq d_B(y) \dot{-} d(a, y) \\
&= d_B(y) \dot{-} d_{A\cap K_{i+1}}(y) \\
&= \sup_{x\in K_i} |d_B(x) \dot{-} d_{A\cap K_{i+1}}(x)| \\
&= |d_B \dot{-} d_{A\cap K_{i+1}}|_{K_i}
\end{aligned}$$

□

We will prove that the lower and upper topology induced by $d'_\mathcal{A}$ is the lower and upper Fell topology, respectively. The *lower Fell topology* has as a subbase all sets

$$U^- := \{A \in \mathcal{A}(X) : A \cap U \neq \emptyset\},\ U \subseteq X \text{ open}$$

and the *upper Fell topology* has as a base all sets

$$(K^c)^+ := \{A \in \mathcal{A}(X) : A \cap K = \emptyset\},\ K \subseteq X \text{ compact}$$

(cf. [Bee93] for the Fell topology).

**Proposition 2.7.4** *If $(X, d)$ is a nice locally compact separable metric space, then the lower and upper topology induced by $d'_{\mathcal{A}}$ coincides with the lower and upper Fell topology, respectively.*

**Proof.** Let $(K_i)_{i\in\mathbb{N}}$ be the exhausting sequence of $X$. We denote by $\tau_<$ and $\tau_>$ the lower and upper topology induced by $d'_{\mathcal{A}}$, and by $B_<(A,\varepsilon)$ and $B_>(A,\varepsilon)$ the corresponding balls, respectively. By $\tau_F^<$ and $\tau_F^>$ we denote the lower and upper Fell topology, respectively. We will prove

$$(1)\ \tau_< \subseteq \tau_F^<, \tau_> \subseteq \tau_F^>, \quad (2)\ \tau_F^< \subseteq \tau_<, \quad (3)\ \tau_F^> \subseteq \tau_>$$

in the following.

(1) Let $A \in \mathcal{A}(X)$ and $\varepsilon > 0$. It suffices to prove that there are sets $V \in \tau_F^<$, $W \in \tau_F^>$ such that $A \in V \subseteq B_<(A,\varepsilon)$ and $A \in W \subseteq B_>(A,\varepsilon)$. Let $\delta := \varepsilon/3$ and $j \in \mathbb{N}$ such that $2^{-j-1} < \delta$ and $A \cap K_{j+1} \neq \emptyset$. Since $A \cap K_{j+1}$ is compact, the open covering $\{B(x,\delta) : x \in A \cap K_{j+1}\}$ admits a finite subcovering $\{B(x_0,\delta), ..., B(x_n,\delta)\}$ with $x_0, ..., x_n \in A \cap K_{j+1}$. Let $U := \bigcup_{k=0}^n B(x_k,\delta)$. Then $K_{j+1} \setminus U = K_{j+1} \cap U^c$ is compact and $W := (K_{j+1}^c \cup U)^+ \in \tau_F^>$. Moreover, $V := \bigcap_{k=0}^n B(x_k,\delta)^- \in \tau_F^<$. Obviously, $A \in V \cap W$. Let $E := \{x_0, ..., x_n\}$. For arbitrary sets $B \in V$, $C \in W$ we obtain $B \cap B(x,\delta) \neq \emptyset$ for all $x \in E$ and $C \cap K_{j+1} \subseteq U = \bigcup_{x\in E} B(x,\delta)$. Thus,

$$d_B(x) \le d_E(x) + \delta \text{ and } d_E(x) \le d_{C\cap K_{j+1}}(x) + \delta$$

for all $x \in X$. Since $d$ is nice, $d_{C\cap K_{j+1}}(x) = d_C(x)$ for all $x \in K_j$. We obtain $|d_B \dot{-} d_C|_{K_j} \le 2\delta$ and thus, in particular, $|d_B \dot{-} d_A|_{K_j} \le 2\delta$ and $|d_A \dot{-} d_C|_{K_j} \le 2\delta$. Hence,

$$d'_{\mathcal{A}}(A,B) \le 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} |d_B \dot{-} d_A|_{K_i} < \delta + 2\delta = \varepsilon$$

and analogously $d'_{\mathcal{A}}(C,A) < \varepsilon$. Thus, $B \in B_<(A,\varepsilon)$, $C \in B_>(A,\varepsilon)$ and hence $V \subseteq B_<(A,\varepsilon)$, $W \subseteq B_>(A,\varepsilon)$.

(2) Let $U \subseteq X$ be open and $A \in U^-$. It suffices to find an $\varepsilon > 0$ such that $B_<(A,\varepsilon) \subseteq U^-$. Since $A \cap U \neq \emptyset$ there is an $x \in A \cap U$ and some $\delta > 0$ such that $B(x,\delta) \subseteq U$. Furthermore, there is some $j \in \mathbb{N}$ such that $x \in K_j$. Let $\varepsilon := 2^{-j-1}\delta$. For each $B \in B_<(A,\varepsilon)$ we obtain $\sum_{i=0}^{\infty} 2^{-i-1} |d_B \dot{-} d_A|_{K_i} = d'_{\mathcal{A}}(A,B) < \varepsilon$. Hence, $|d_B \dot{-} d_A|_{K_j} < \delta$. Since $d_A(x) = 0$ and $x \in K_j$, this implies $d_B(x) < \delta$. Consequently, $B \cap B(x,\delta) \neq \emptyset$ and $B \in U^-$, i.e. $B_<(A,\varepsilon) \subseteq U^-$.

(3) Let $K \subseteq X$ be compact and $A \in (K^c)^+$. It suffices to find an $\varepsilon > 0$ such that $B_>(A, \varepsilon) \subseteq (K^c)^+$. Now $\delta := \inf_{x \in K} d_A(x) > 0$ since $K \subseteq A^c$. Let $j \in \mathbb{N}$ be such that $K \subseteq K_j^\circ$ and let $\varepsilon := 2^{-j-1}\delta$. For each $C \in B_>(A, \varepsilon)$ we obtain $\sum_{i=0}^{\infty} 2^{-i-1} |d_A \dot{-} d_C|_{K_i} = d'_{\mathcal{A}}(C, A) < \varepsilon$. Hence, $|d_A \dot{-} d_C|_K \leq |d_A \dot{-} d_C|_{K_j} < \delta$. Thus, $\inf_{x \in K} d_C(x) > 0$ by definition of $\delta$ and thus $K \subseteq C^c$, i.e. $C \in (K^c)^+$ and $B_>(A, \varepsilon) \subseteq (K^c)^+$.

□

The metric $(d'_{\mathcal{A}})_\star$, associated with $d'_{\mathcal{A}}$, is equivalent to the metric $d_{\mathcal{A}}$, defined by

$$d_{\mathcal{A}} : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}, (A, B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} |d_B - d_A|_{K_i},$$

since

$$(d'_{\mathcal{A}})_\star \leq d_{\mathcal{A}} \leq 2(d'_{\mathcal{A}})_\star. \tag{2.2}$$

The previous result especially shows that $d_{\mathcal{A}}$ is compatible with the Fell topology. In case that $X$ is compact, $\mathcal{A}(X) = \mathcal{K}(X)$ and the quasi-metrics $d'_{\mathcal{A}}$ and $d'_{\mathcal{K}}$ are uniformly equivalent, as the following lemma shows.

**Lemma 2.7.5** *If $(X, d)$ is a compact separable metric space, then there is a constant $c \in \mathbb{N}$ such that*

$$d'_{\mathcal{A}}(A, B) \leq d'_{\mathcal{K}}(A, B) \leq c \cdot d'_{\mathcal{A}}(A, B)$$

*for all $A, B \in \mathcal{A}(X) = \mathcal{K}(X)$.*

**Proof.** Let $(K_i)_{i \in \mathbb{N}}$ be the exhausting sequence of $X$. Then $(K_i^\circ)_{i \in \mathbb{N}}$ is an open covering of $X$ and if $X$ is compact, then there is some $j \in \mathbb{N}$ such that $K_i = X$ for all $i \geq j$. Let $c := 2^j$. Then

$$d'_{\mathcal{A}}(A, B) = \sum_{i=0}^{\infty} 2^{-i-1} |d_B \dot{-} d_A|_{K_i} \leq |d_B \dot{-} d_A|_{K_j} = d'_{\mathcal{K}}(A, B)$$

and

$$\begin{aligned} d'_{\mathcal{K}}(A, B) &= \sup_{a \in A} d_B(a) = |d_B \dot{-} d_A|_X = \sum_{i=0}^{\infty} 2^{-i-1} |d_B \dot{-} d_A|_{K_{i+j}} \\ &\leq \sum_{i=0}^{\infty} 2^{-i+j-1} |d_B \dot{-} d_A|_{K_i} \\ &= c \cdot d'_{\mathcal{A}}(A, B). \end{aligned}$$

□

Similarly, $x \in K_j$ implies

$$d'_{\mathcal{A}}(\{x\}, B) \leq d_B(x) \leq 2^j \cdot d'_{\mathcal{A}}(\{x\}, B) \qquad (2.3)$$

for each set $B \in \mathcal{A}(X)$. The following lemma will be useful for some later results.

**Lemma 2.7.6** *Let $(X, d)$ be a nice locally compact metric space. Then for all $A, B \in \mathcal{A}(X)$ we obtain $d'_{\mathcal{A}}(A \cup B, B) = d'_{\mathcal{A}}(A, B)$.*

**Proof.** Let $C := A \cup B$. We prove $d'_{\mathcal{A}}(C, B) = d'_{\mathcal{A}}(A, B)$. Since $A \subseteq C$, we obtain "$\geq$" since $d'_{\mathcal{A}}$ is isotone. Let us assume $d'_{\mathcal{A}}(C, B) > d'_{\mathcal{A}}(A, B)$. Then there is some $i \in \mathbb{N}$ such that

$$|d_B \dot{-} d_C|_{K_i} > |d_B \dot{-} d_A|_{K_i} \quad (*)$$

and some $x \in K_i$ such that $d_B(x) \dot{-} d_C(x) = \sup_{y \in K_i} |d_B(y) \dot{-} d_C(y)|$. Moreover, we obtain $d_C(x) \leq d_A(x)$ since $A \subseteq C$. Let us assume $d_C(x) < d_A(x)$. Then there is some $y \in C$ with $d(x, y) < d_A(x)$ and for all such $y \in C$ we obtain $y \notin A$ and thus $y \in B$. Consequently, $d_C(x) = d_B(x)$. But this implies $|d_B \dot{-} d_C|_{K_i} = 0$ which contradicts $(*)$. Thus, $d_C(x) = d_A(x)$ and $|d_B \dot{-} d_A|_{K_i} \geq |d_B \dot{-} d_C|_{K_i}$ which also contradicts $(*)$. Thus, $d'_{\mathcal{A}}(C, B) = d'_{\mathcal{A}}(A, B)$. □

The following theorem summarizes some topological properties of the quasi-metric space $(\mathcal{A}(X), d'_{\mathcal{A}})$ and its conjugate.

**Theorem 2.7.7** *If $(X, d)$ is a nice locally compact separable metric space, then the quasi-metric space $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}})$ is upper generated and continuous from above and $\mathcal{A}(X)$ is strongly dense in $(\mathcal{A}(X), d'_{\mathcal{A}})$. The same holds for the conjugate quasi-metric space $(\mathcal{A}(X), \mathcal{A}(X), \overline{d'_{\mathcal{A}}})$.*

**Proof.** Obviously, $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}})$ is lower and upper generated.

We start to prove that $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}})$ is consistent from below with respect to the lower topology. Let $(A_n)_{n \in \mathbb{N}}$ be an increasing chain in $\mathcal{A}(X)$, i.e. $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and let $A := \sup_{n \in \mathbb{N}} A_n = \overline{\bigcup_{n=0}^{\infty} A_n}$. By Proposition 2.7.4 it suffices to prove that $(A_n)_{n \in \mathbb{N}}$ converges to $A$ with respect to the lower Fell topology. Let $U_0, ..., U_m \subseteq X$ be open subsets such that

$$A \in U_0^- \cap ... \cap U_m^- =: U,$$

i.e. $\overline{\bigcup_{n=0}^{\infty} A_n} \cap U_i \neq \emptyset$ for $i = 0, ..., m$. Then there is some $k_i \in \mathbb{N}$ for each $i = 0, ..., m$ such that $A_{k_i} \cap U_i \neq \emptyset$. Since $A_n \subseteq A_{n+1}$ we obtain $A_n \cap U_i \neq \emptyset$ for all $n \geq k := \max\{k_0, ..., k_m\}$ and $i = 0, ..., m$. Hence, $A_n \in U$ for all $n \geq k$, i.e. $(A_n)_{n \in \mathbb{N}}$ converges to $A$ with respect to the lower topology induced by $d'_{\mathcal{A}}$.

Now we prove that $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}})$ is consistent from above with respect to the upper topology. Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing chain in $\mathcal{A}(X)$, i.e. $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$, such that $A := \inf_{n\in\mathbb{N}} A_n = \bigcap_{n=0}^{\infty} A_n \neq \emptyset$. By Proposition 2.7.4 it suffices to prove that $(A_n)_{n\in\mathbb{N}}$ converges to $A$ with respect to the upper Fell topology. Let $K \subseteq X$ be compact such that $A \in (K^c)^+$, i.e. $K \subseteq A^c = \bigcup_{n=0}^{\infty} A_n^c$. Since $K$ is compact and $\{A_n^c : n \in \mathbb{N}\}$ is an open covering of $K$, there is some finite subcovering, i.e. there is some $k \in \mathbb{N}$ such that $K \subseteq \bigcup_{n=0}^{k} A_n^c$. Since $A_{n+1} \subseteq A_n$ we obtain $K \subseteq A_n^c$ for all $n \geq k$. Thus $A_n \in (K^c)^+$ for all $n \geq k$, i.e. $(A_n)_{n\in\mathbb{N}}$ converges to $A$ with respect to the upper topology induced by $d'_{\mathcal{A}}$.

By Proposition 2.3.4 it follows that $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}})$ is continuous from above. Obviously, the conjugate space $(\mathcal{A}(X), \mathcal{A}(X), \overline{d'_{\mathcal{A}}})$ is upper generated, by the proof above it follows that the space is consistent from above and by Proposition 2.3.4 it follows that it is continuous from above.

Now, we prove that $\mathcal{A}(X)$ is strongly dense in $(\mathcal{A}(X), d'_{\mathcal{A}})$ with constant $c := 1$. Let $A, B \in \mathcal{A}(X)$ and $\varepsilon > 0$. Then $C := A \cup B \in \mathcal{A}(X)$ and we obtain $d'_{\mathcal{A}}(A, C) = 0 < \varepsilon$ and hence by Lemma 2.7.6 we can conclude $(d'_{\mathcal{A}})_\star(C, B) = d'_{\mathcal{A}}(C, B) = d'_{\mathcal{A}}(A, B) < c \cdot d'_{\mathcal{A}}(A, B) + \varepsilon$.

Finally, we prove that $\mathcal{A}(X)$ is strongly dense in $(\mathcal{A}(X), \overline{d'_{\mathcal{A}}})$. In case that $X$ is compact the claim follows from Lemma 2.7.5 and Theorem 2.5.2. Thus, let us assume that $X$ is locally compact with exhausting sequence $(K_i)_{i\in\mathbb{N}}$ but not compact and hence we can assume without loss of generality $K_{i+1} \setminus K_i \neq \emptyset$ for all $i \in \mathbb{N}$. Let $c := 8$, $A, B \in \mathcal{A}(X)$ and $\varepsilon > 0$. Then there is some $j \in \mathbb{N}$ such that $2^{-j-1} < \varepsilon/2$. We construct a set $C := \overline{\{a_b : b \in B\}} \in \mathcal{A}(X)$ with an element $a_b \in X$ for each $b \in B$. Let $b \in B$. Then there is some $i_b \in \mathbb{N}$ such that $b \in K_{i_b} \setminus K_{i_b-1}$ (we define $K_{-1} := K_{-2} := \emptyset$).

<u>1. Case.</u> If there is some $a \in A \setminus K_{i_b-2}$ such that $d(a, b) \leq d_A(b) + \varepsilon/4$, then let $a_b := a$.

<u>2. Case.</u> Otherwise, there is some $a \in A \cap K_{i_b-2}$ such that $d(a, b) \leq d_A(b) + \varepsilon/4$ and since $d$ is nice, we obtain $d(a, b) = 1$. In this case let $a_b \in K_{i_b+j+3} \setminus K_{i_b+j+2}$. Since $d$ is nice we obtain $d(a_b, b) = 1$ too.

Thus we have chosen some $a_b \in X$ for each $b \in B$ with $d(a_b, b) \leq d_A(b) + \varepsilon/4$. Since $C \cap K_{j+1} \subseteq A$ and $d$ is nice we immediately obtain by Lemma 2.7.3

$$\overline{d'_{\mathcal{A}}}(A, C) \leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} |d_A \dot{-} d_{C\cap K_{j+1}}|_{K_i} = 2^{-j-1} < \varepsilon.$$

(Here, we tacitly assume $d_\emptyset = 1$.) Now we claim

$$|d_C - d_B|_{K_i} \leq |d_A \dot{-} d_B|_{K_{i+3}} + \frac{\varepsilon}{2} \quad (*)$$

for all $i \leq j$. Let $i \leq j$. If $|d_C \dot{-} d_B|_{K_i} = 0$, then $|d_C \dot{-} d_B|_{K_i} \leq |d_A \dot{-} d_B|_{K_{i+3}}$ holds; otherwise $d_B(x) < 1$ for some $x \in K_i$ and thus $B \cap K_{i+1} \neq \emptyset$, since $d$ is nice. Hence, by Lemma 2.7.3

$$\begin{aligned}
|d_C \dot{-} d_B|_{K_i} &\leq \sup_{b \in B \cap K_{i+1}} d_C(b) \\
&\leq \sup_{b \in B \cap K_{i+1}} d(a_b, b) \\
&\leq \sup_{b \in B \cap K_{i+1}} d_A(b) + \frac{\varepsilon}{4} \\
&\leq |d_A \dot{-} d_{B \cap K_{i+1}}|_{K_{i+1}} + \frac{\varepsilon}{4} \\
&\leq |d_A \dot{-} d_B|_{K_{i+3}} + \frac{\varepsilon}{2}
\end{aligned}$$

On the other hand, if $|d_B \dot{-} d_C|_{K_i} = 0$, then $|d_B \dot{-} d_C|_{K_i} \leq |d_A \dot{-} d_B|_{K_{i+3}}$ holds; otherwise $d_C(x) < 1$ for some $x \in K_i$ and thus $C \cap K_{i+1} \neq \emptyset$, since $d$ is nice. Hence by Lemma 2.7.3, there is some $y \in C \cap K_{i+1}$ such that $|d_B \dot{-} d_C|_{K_i} \leq \sup_{x \in C \cap K_{i+1}} d_B(x) = d_B(y)$ and there is some $b \in B$ such that $d(y, a_b) \leq \varepsilon/4$ and $a_b \in K_{i+2}$. Since $i \leq j$, we can conclude $a_b \in A \setminus K_{i_b - 2}$ by construction of $C$. Thus, $i_b < i + 4$, i.e. $b \in K_{i+3}$, and consequently

$$\begin{aligned}
|d_B \dot{-} d_C|_{K_i} &\leq d_B(y) \\
&\leq d_B(a_b) + \frac{\varepsilon}{4} \\
&\leq d(a_b, b) + \frac{\varepsilon}{4} \\
&= d_A(b) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&\leq |d_A \dot{-} d_B|_{K_{i+3}} + \frac{\varepsilon}{2}.
\end{aligned}$$

Altogether, this proves $(*)$ and we can conclude

$$\begin{aligned}
(\overline{d'_{\mathcal{A}}})_\star(C, B) &\leq d_{\mathcal{A}}(C, B) \\
&\leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} |d_C - d_B|_{K_i} \\
&\leq 2^{-j-1} + \sum_{i=0}^{j} 2^{-i-1} \left( |d_A \dot{-} d_B|_{K_{i+3}} + \frac{\varepsilon}{2} \right) \\
&\leq 2^{-j-1} + \frac{\varepsilon}{2} + 8 \cdot \sum_{i=0}^{j+3} 2^{-i-1} |d_A \dot{-} d_B|_{K_i} \\
&< c \cdot \overline{d'_{\mathcal{A}}}(A, B) + \varepsilon
\end{aligned}$$

□

# Chapter 3

# Recursive Quasi-Metric Spaces

## 3.1 Introduction

The purpose of this chapter is to develop a reasonable notion of a recursive quasi-metric space in analogy to the notion of a recursive metric space. The study of recursive metric spaces has a long tradition in computable analysis: Lacombe has investigated recursive complete separable metric spaces [Lac59] from a classical point of view, in the russian school of constructive analysis Ceitin [Ceĭ62], Šanin [Šan68], and Kušner [Kuš84] have investigated metric spaces. Similarly, Moschovakis [Mos64] and, later on, Spreen [Spr98] have investigated recursive metric spaces restricted to computable points. In type-2 theory of effectivity Weihrauch [Wei93, Wei00] studied computable metric spaces, in the domain representation approach Blanck investigated them [Bla97] and the Pour-El and Richards approach to computable analysis has been extended to metric spaces by Mori, Tsujii, Yasugi and Washihara [YMT99, WY96]. In [Bra02a, Bra99a] we have shown that recursive metric spaces can be used to construct natural data structures for computations in metric spaces. All the definitions used for recursive/computable metric spaces lead to similar or even equivalent concepts. We will use the following definition.

**Definition 3.1.1 (Recursive metric space)** We will call a triple $(X, d, \alpha)$ a *recursive metric space*, if

(1) $d : X \times X \to \mathbb{R}$ is a metric on $X$,

(2) $\alpha : \mathbb{N} \to X$ is dense in $X$,

(3) $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence of real numbers.

If $(X, d, \alpha)$ fulfills at least (1) and (2), then it is called a *separable metric space*.

Here we assume that the reader is familiar with the notion of a computable sequence of real numbers in sense of computable analysis (cf. for instance [PER89, Wei00]). A sequence $\alpha : \mathbb{N} \to X$ is called *dense* in $X$, if the range of the sequence $\alpha$ is dense in $(X, d)$.

The basic idea of a recursive metric space is that a given notion of computability on a countable set (induced by $\alpha$) is "lifted" to the whole space in a way such that the metric as well as the limit operation becomes computable. Here, by the *limit operation* we mean the operation

$$\mathrm{Lim} :\subseteq X^{\mathbb{N}} \to X, (x_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} x_n,$$

restricted to rapidly converging sequences:

$$\mathrm{dom}(\mathrm{Lim}) := \{(x_n)_{n\in\mathbb{N}} : (\forall n > k)\ d(x_n, x_k) \leq 2^{-k} \text{ and } (x_n)_{n\in\mathbb{N}} \text{ converges}\}.$$

In the following we will briefly recall some fundamental concepts of the representation based approach to computable analysis. For details we refer the reader to Weihrauch [Wei00]. The basic idea of computable analysis is to call a (possibly partial) function $f :\subseteq X \to Y$ *computable*, if there exists a Turing machine which transfers each infinite sequence $p \in \mathbb{N}^{\mathbb{N}}$ that represents some input $x \in X$ into some sequence $q \in \mathbb{N}^{\mathbb{N}}$ which represents the result $f(x)$. Of course, such a Turing machine has to compute infinitely long, but in the long run each infinite input sequence is transformed into an appropriate output sequence. It is a reasonable restriction that only Turing machines with one-way output tape are allowed (because otherwise the output after some finite time would be useless, since it could be changed by the machine later on). More

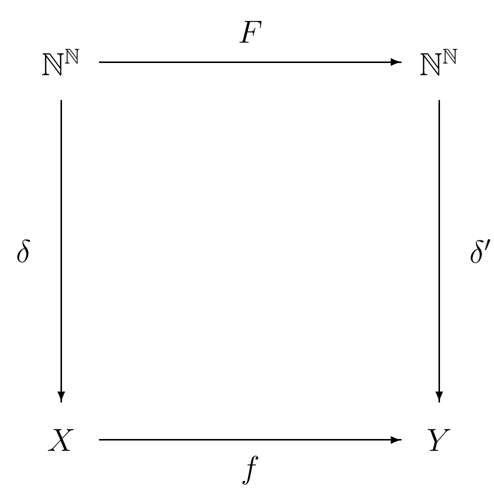


Figure 3.1: Computability with respect to representations

formally, a *representation* of a set $X$ is a surjective mapping $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

Using this notion we can define computable functions precisely. Figure 3.1 illustrates the situation.

**Definition 3.1.2 (Computable functions)** Let $\delta$ and $\delta'$ be representations of $X$ and $Y$, respectively. A function $f :\subseteq X \to Y$ is called $(\delta, \delta')$*–computable*, if there exists a Turing machine $M$ such that $f\delta(p) = \delta' F_M(p)$ for all sequences $p \in \mathrm{dom}(f\delta)$.

Here, $F_M :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ denotes the function computed by Turing machine $M$. We can define a corresponding notion of $(\delta, \delta')$*–continuity*, if we replace the computable function $F_M$ by a continuous function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. Here, the *Baire space* $\mathbb{N}^\mathbb{N}$ is endowed with the product topology of the discrete topology on $\mathbb{N}$. Whenever we have representations $\delta$ and $\delta'$ of sets $X$ and $Y$, respectively, we can canonically define a representation $[\delta, \delta']$ of the product $X \times Y$ and a representation $\delta^\infty$ of the infinite product $X^\mathbb{N}$ by

$$[\delta, \delta']\langle p, q\rangle := (\delta(p), \delta'(q))$$

and

$$\delta^\infty\langle p_0, p_1, ...\rangle := (\delta(p_0), \delta(p_1), ...),$$

where we define the pairing functions

$$\begin{aligned} \langle p, q\rangle(k) &:= \begin{cases} p(n) & \text{if} \quad k = 2n \\ q(n) & \text{if} \quad k = 2n+1, \end{cases} \\ \langle p_0, p_1, ...\rangle\langle n, k\rangle &:= p_n(k) \end{aligned}$$

for all $p, q, p_i \in \mathbb{N}^\mathbb{N}, i, k, n \in \mathbb{N}$ with the help of *Cantor tuples*

$$\langle n, k\rangle := \frac{1}{2}(n+k)(n+k+1) + k.$$

Analogously, one can define $[\delta_1, ..., \delta_n]$ for $n > 2$. With each separable metric space we can canonically associate its *Cauchy representation*.

**Definition 3.1.3 (Cauchy representation)** Let $(X, d, \alpha)$ be a separable metric space. Then the *Cauchy representation* of $X$ is defined by $\delta_X := \mathrm{Lim} \circ \alpha^\mathbb{N}$, i.e.

$$\delta_X(p) = \lim_{k\to\infty} \alpha p(k)$$

for all $p \in \mathbb{N}^\mathbb{N}$ such that $(\alpha p(k))_{k\in\mathbb{N}} \in \mathrm{dom}(\mathrm{Lim})$.

Roughly speaking, $\delta_X(p) = x$, if $p$ encodes a Cauchy sequence which rapidly converges to $x$. Occasionally, we will also use some standard representation $\delta_\mathbb{N}$ of the natural numbers $\mathbb{N} := \{0, 1, 2, ...\}$. Given two representations $\delta, \delta'$ of a set $X$, we will say that $\delta$ is *reducible* to $\delta'$, if there is some computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $\delta(p) = \delta' F(p)$ (or, equivalently, if the identity is $(\delta, \delta')$–computable). In this situation we write $\delta \leq \delta'$. We will say that $\delta$ and $\delta'$ are *equivalent*, if $\delta \leq \delta'$, as well as $\delta' \leq \delta$ holds. In this case we write $\delta \equiv \delta'$. We can define a corresponding concept of continuous reducibility, if we replace computable functions $F$ by continuous functions. In this case we will write $\delta \leq_{\mathrm{t}} \delta'$ and $\delta \equiv_{\mathrm{t}} \delta'$ for the corresponding topological reducibility and equivalence. The following result, which has been proved in [Bra02a, Bra99a], characterizes the equivalence class of the Cauchy representation in terms of the metric and the limit operation.

**Proposition 3.1.4 (Characterization of the Cauchy representation)** *Let $(X, d, \alpha)$ be a recursive metric space with Cauchy representation $\delta_X$ and let $\delta$ be a further representation of $X$. Then*

*(1)* $\delta \leq \delta_X \iff d : X \times X \to \mathbb{R}$ *is* $([\delta, \delta_X], \delta_{\mathbb{R}})$*–computable,*

*(2)* $\delta_X \leq \delta \iff \mathrm{Lim} :\subseteq X^{\mathbb{N}} \to X$ *is* $(\delta_X^\infty, \delta)$*–computable.*

The Cauchy representation of a recursive metric space has some nice properties. Especially, it is an *admissible* representation in the sense of Weihrauch [Wei00]. Admissible representations are those representations which are maximal among all continuous representations with respect to continuous reducibility.

**Definition 3.1.5 (Admissibility)** A representation $\delta$ of a set $X$ is called *admissible* with respect to a topology $\tau$ on $X$, if $\delta$ is continuous with respect to $\tau$ and $\delta' \leq_{\mathrm{t}} \delta$ holds for all representations $\delta'$ of $X$ which are continuous with respect to $\tau$.

In [BH02] it has been proved that, essentially, a representation is admissible with respect to a given $T_0$–topology with countable base, if it is continuous and if it admits a surjective and open restriction. The most important property of admissible representations is that they fit together with the given topologies. The so-called *Main Theorem of Type-2 Theory of Effectivity* states that a function $f :\subseteq X \to Y$ is $(\delta, \delta')$–continuous, if and only if it is continuous, provided that $\delta, \delta'$ are admissible representations of $X, X'$ with respect to the corresponding $T_0$-topologies with countable bases [Wei00]. Especially, each $(\delta, \delta')$–computable function is continuous in this case. We mention some standard examples of recursive metric spaces.

**Example 3.1.6 (Recursive metric spaces)**

*(1)* $(\mathbb{R}^n, d_{\mathbb{R}^n}, \alpha_{\mathbb{R}^n})$ *with the Euclidean metric*

$$d_{\mathbb{R}^n}(x,y) := \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

*and some standard enumeration* $\alpha_{\mathbb{R}^n}$ *of all rational points* $\mathbb{Q}^n$ *is a recursive metric space. The computable points in this space are exactly the computable points* $x \in \mathbb{R}^n$*.*

*(2)* $(\mathcal{K}(\mathbb{R}^n), d_{\mathcal{K}}, \alpha_{\mathcal{K}})$ *with the set* $\mathcal{K}(\mathbb{R}^n)$ *of non-empty compact subsets of* $\mathbb{R}^n$ *and the Hausdorff metric*

$$d_{\mathcal{K}}(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d_{\mathbb{R}^n}(a,b), \sup_{b\in B}\inf_{a\in A} d_{\mathbb{R}^n}(a,b)\right\}$$

*and some standard numbering* $\alpha_{\mathcal{K}}$ *of the non-empty finite subsets of* $\mathbb{Q}^n$ *is a recursive metric space. The computable points in this space are exactly the recursive compact sets* $A \subseteq \mathbb{R}^n$ *(cf. [BW99]).*

*(3)* $(\mathcal{C}(\mathbb{R}^n), d_{\mathcal{C}}, \alpha_{\mathcal{C}})$ *with the set* $\mathcal{C}(\mathbb{R}^n)$ *of continuous functions* $f : \mathbb{R}^n \to \mathbb{R}$ *and the metric*

$$d_{\mathcal{C}}(f,g) := \sum_{i=0}^{\infty} 2^{-i-1} \frac{\sup_{x\in[-i,i]^n} |f(x)-g(x)|}{1+\sup_{x\in[-i,i]^n} |f(x)-g(x)|}$$

*and some standard numbering* $\alpha_{\mathcal{C}}$ *of the rational polynomials* $\mathbb{Q}[x_1, ..., x_n]$ *is a recursive metric space. The computable points in this space are exactly the computable functions* $f : \mathbb{R}^n \to \mathbb{R}$ *(cf. [Bra99a]).*

Here, a point $x \in X$ of a recursive metric space $(X, d, \alpha)$ with Cauchy representation $\delta_X$ is called *computable*, if there exists some computable $p$ such that $\delta_X(p) = x$. It should be mentioned that there are other computable objects in analysis, such as left- or right computable real numbers, r.e. or co-r.e. compact subsets, lower- or upper semi-computable functions (cf. [Wei00]), which cannot be considered as computable points of recursive metric spaces in the same sense as it has been demonstrated for other computable objects in the previous example. The reason is that these other computable objects naturally have to be considered as points of topological spaces which are asymmetric and thus not metrizable. However, we will see that all mentioned objects can be considered as computable points of suitably recursive quasi-metric spaces.

We close this introduction with a survey of the organization of the chapter. In the following Section 2 we will introduce the notion of a semi-recursive quasi-metric space. In Section 3 we prove that the induced Dedekind representations of semi-recursive quasi-metric spaces are admissible with respect to the weak upper topology. In Section 4 we discuss conditions which are helpful to prove that certain concrete examples of quasi-metric spaces are semi-recursive. In Section 5 we investigate how our computability concepts for quasi-metric spaces can be related to the computability concept on metric spaces. In Section 6 we will discuss an effective generation property of quasi-metric spaces which is another helpful tool to prove that certain concrete spaces are semi-recursive. In Sections 7, 8 and 9 we will introduce examples of semi-recursive quasi-metric spaces for the hyperspace of compact subsets, the spaces of upper and lower semi-continuous functions and the hyperspace of closed subsets.

## 3.2 Semi-recursive quasi-metric spaces

In this section we will introduce the concept of a semi-recursive quasi-metric space. We recall that $(X, Y, d)$ is an upper generated quasi-metric space, if and only if $(X, d)$ is a quasi-metric space, $Y \subseteq X$ and

$$\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X, (x_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} x_n$$

is surjective. A corresponding property holds for lower generated spaces with sup instead of inf. Typically, we will endow the subset $Y$ with the associated metric $d_\star$ and in all our applications $(Y, d_\star)$ will be separable. Figure 3.2 illustrates the situation of a quasi-metric space which is generated by a separable metric space. For each quasi-metric space which is lower or upper generated

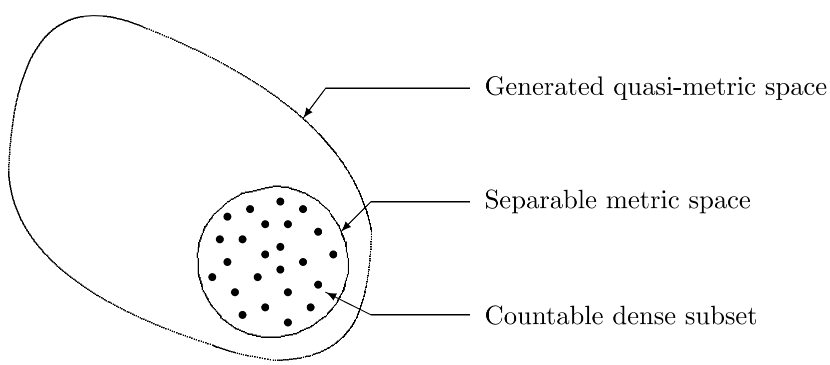


Figure 3.2: Generated quasi-metric spaces

by a separable metric space we can define a corresponding Dedekind representation, since Sup and Inf are surjective, respectively.

**Definition 3.2.1 (Dedekind representation)** Let $(X, Y, d)$ be an upper generated quasi-metric space and let $(Y, d_\star, \alpha)$ be a separable metric space with Cauchy representation $\delta_Y$. Then the *upper Dedekind representation* is defined by

$$\delta_{X_>} := \mathrm{Inf} \circ \delta_Y^\infty.$$

Analogously, we define the *lower Dedekind representation* for lower generated quasi-metric spaces by $\delta_{X_<} := \mathrm{Sup} \circ \delta_Y^\infty$.

As a standard example of an upper and lower generated quasi-metric spaces we have seen $(\mathbb{R}, \mathbb{R}, d)$ with $d(x, y) := x \dot{-} y$. It is easy to see that the corresponding Dedekind representations $\delta_{\mathbb{R}_<}$ and $\delta_{\mathbb{R}_>}$ are admissible representations of $\mathbb{R}_<$ and $\mathbb{R}_>$, respectively (typically called $\rho_<$ and $\rho_>$, cf. [Wei00]). Now we are prepared to define semi-recursive quasi-metric spaces. The basic idea is to "lift" the notion of computability from the associated metric subspace $(Y, d_\star)$ to the generated quasi-metric space $(X, Y, d)$.

**Definition 3.2.2 (Semi-recursive quasi-metric spaces)** We call a tuple $(X, Y, d, \alpha)$ a *semi-recursive quasi-metric space*, if the following applies:

(1) $(X, Y, d)$ is an upper generated quasi-metric space with upper Dedekind representation $\delta_{X_>}$,

(2) $(Y, d_\star|_{Y\times Y}, \alpha)$ is a recursive metric space with Cauchy representation $\delta_Y$,

(3) $d|_{X\times Y} : X \times Y \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–computable,

(4) $\delta \le \delta_{X_>}$ for each representation $\delta$ of $X$ such that $d|_{X\times Y} : X \times Y \to \mathbb{R}_>$ is $([\delta, \delta_Y], \delta_{\mathbb{R}_>})$–computable.

We will call $(X, Y, d, \alpha)$ a *recursive quasi-metric space*, if $(X, Y, d, \alpha)$ as well as the conjugate space $(X, Y, \overline{d}, \alpha)$ are semi-recursive quasi-metric spaces.

By abuse of notation we will sometimes say that $(X, Y, d)$ is a semi-recursive quasi-metric space and occasionally we will simply write $d_\star$ instead of $d_\star|_{Y\times Y}$ and $d$ instead of $d|_{X\times Y}$.

Analogously as in Definition 3.1.1 of a recursive metric space (1) and (2) express the fact that we already have some notion of computability on a subset (on the generating set in the quasi-metric case and on a dense subset in the metric case) and (3) expresses the fact that the distance function is computable in a certain sense. Property (4) has no counterpart in case of recursive metric spaces; Proposition 3.1.4 shows that the corresponding property for metric spaces is fulfilled automatically. One could express this difference by the statement that recursive metric spaces are always "effectively separable" while semi-recursive quasi-metric spaces are not "effectively generated" in general. But we will not make these statements precise. Later on, we will see that some additional purely topological properties on the quasi-metric also guarantee that (1) to (3) imply (4). A standard example of a recursive quasi-metric space is the following.

**Example 3.2.3** *The space* $(\mathbb{R}, \mathbb{R}, d, \alpha_{\mathbb{R}})$ *with* $d(x,y) := x \dot{-} y$ *is a recursive quasi-metric space.*

**Proof.** By Example 2.2.8 $(\mathbb{R}, \mathbb{R}, d)$ is an upper generated quasi-metric space with upper Dedekind representation $\delta_{\mathbb{R}_>}$. By Example 3.1.6 $(\mathbb{R}, d_\star, \alpha_{\mathbb{R}})$ is a recursive metric space with Cauchy representation $\delta_{\mathbb{R}}$. Now it is easy to see that $d$ is $([\delta_{\mathbb{R}_>}, \delta_{\mathbb{R}}], \delta_{\mathbb{R}_>})$–computable. This follows directly from

$$\inf_{n\in\mathbb{N}}(x_n \dot{-} y) = \left(\inf_{n\in\mathbb{N}} x_n\right) \dot{-} y$$

for all sequences $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $y \in \mathbb{R}$. Now let $\delta$ be some representation of $\mathbb{R}$ such that $d$ is $([\delta, \delta_{\mathbb{R}}], \delta_{\mathbb{R}_>})$–computable. Then given some $p \in \mathrm{dom}(\delta)$ and $n \in \mathbb{N}$ we can compute all rational upper bounds

$$r > d(\delta(p), \alpha_{\mathbb{R}}(n)) = \delta(p) \dot{-} \alpha_{\mathbb{R}}(n).$$

But in this case $r + \alpha_{\mathbb{R}}(n) > \delta(p)$ and by repeating this process for arbitrary $n$ we obtain arbitrary good rational upper bounds of $\delta(p)$. Hence $\delta \leq \delta_{\mathbb{R}_>}$. $\square$

It should be mentioned that one could introduce an analog concept of a "lower" semi-recursive quasi-metric space by replacing the word "upper" by "lower" in condition (1) of Definition 3.2.2 and by replacing $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–computability of $d$ by $([\delta_{X_<}, \delta_Y], \delta_{\mathbb{R}_<})$–computability in (3) and correspondingly in (4). Some but not all parts of the theory could be dualized to these spaces. Altogether this variant seems to be less fruitful. An intuitive reason for this non-duality is that from the topological as well as from the computational point of view upper bounds on distances are more important and useful than lower bounds.

Now we will prove a characterization of the Dedekind representation which corresponds to the characterization of the Cauchy representation in Proposition 3.1.4. In this case the operation Inf can be used to "synthesize" and the quasi-metric $d$ can be used to "analyze" a quasi-metric space. The proof follows directly from the definition of a semi-recursive quasi-metric space and from the definition of the Dedekind representations. In a certain sense semi-recursive quasi-metric spaces are just designed to fit to the following proposition.

**Proposition 3.2.4 (Dedekind representations)** *Let $(X, Y, d, \alpha)$ be a semi-recursive quasi-metric space with upper Dedekind representation $\delta_{X_>}$, let $\delta_Y$ be the Cauchy representation of the associated metric space $(Y, d_\star, \alpha)$ and let $\delta$ be a further representation of $X$. Then*

*(1)* $\delta \leq \delta_{X_>} \iff d : X \times Y \to \mathbb{R}_>$ *is* $([\delta, \delta_Y], \delta_{\mathbb{R}_>})$*–computable,*

*(2)* $\delta_{X_>} \leq \delta \iff \mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X$ *is* $(\delta_Y^\infty, \delta)$*–computable.*

**Proof.** (1) follows directly from Definition 3.2.2 (3) and (4) and (2) follows since $\delta_{X_>} = \mathrm{Inf} \circ \delta_Y^\infty$. □

This result can also be considered as a stability theorem which states that the structure of semi-recursive quasi-metric spaces characterizes the corresponding computability theory (cf. [Bra99b, Bra99a, Bra02a]).

**Corollary 3.2.5 (Stability Theorem)** *Let $(X, Y, d, \alpha)$ be a semi-recursive quasi-metric space and let $\delta_Y$ be the Cauchy representation of the associated metric space $(Y, d_\star, \alpha)$. Then the upper Dedekind representation $\delta_{X_>}$ is, up to computable equivalence, the only representation of $X$ such that*

*(1)* $d : X \times Y \to \mathbb{R}_>$ *is* $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$*–computable,*

*(2)* $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X$ *is* $(\delta_Y^\infty, \delta_{X_>})$*–computable.*

Now it is natural to ask whether the equivalence class, characterized in the previous corollary, contains an admissible representation of the corresponding quasi-metric space. We study this question is Section 3.3.

Up to now we only know that the infimum is computable on sequences of the generating space. We close this section with an easy proposition which shows that this operation is also computable on sequences of the generated space.

**Proposition 3.2.6 (Infimum)** *If $(X, Y, d)$ is a semi-recursive quasi-metric space, then*

$$\mathrm{Inf}_> :\subseteq X^{\mathbb{N}} \to X, (x_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} x_n$$

*is $(\delta_{X_>}^\infty, \delta_{X_>})$–computable.*

**Proof.** The statement follows from the fact that

$$\inf_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}} y_{nk} = \inf_{(n,k)\in\mathbb{N}^2} y_{nk},$$

provided that $\inf_{k\in\mathbb{N}} y_{nk}$ exists for all $n \in \mathbb{N}$. □

## 3.3 Admissibility of the Dedekind representation

In this section we want to show that the upper Dedekind representation of a semi-recursive quasi-metric space is admissible. In general the Dedekind representation is not admissible with respect to the upper quasi-metric topology but with respect to the weak upper topology. It is easy to see that the weak upper topology admits a countable base, if $(Y, d_\star)$ is separable. Before we state the main result of this section we formulate a technical lemma.

**Lemma 3.3.1** *Let $(X, Y, d)$ be an upper generated quasi-metric space, let $Z$ be a topological space and $f :\subseteq Z \to X$ a function, let $X$ be endowed with the weak upper topology and let $Y$ be endowed with the metric topology, induced by $d_\star$. Then*

$$g :\subseteq Z \times Y \to \mathbb{R}_>, (z, y) \mapsto d(f(z), y)$$

*is continuous, if and only if $f :\subseteq Z \to X$ is continuous.*

**Proof.** First, we show that $d|_{X\times Y} : X \times Y \to \mathbb{R}_>$ is continuous. Let $r \in \mathbb{R}$ and $(x, y) \in d|_{X\times Y}^{-1}(-\infty, r)$. Then $\varepsilon := d(x, y) < r$ and $\delta := \frac{r-\varepsilon}{2} > 0$. Let $x' \in B_>(y, \varepsilon + \delta)$ and $y' \in B(y, \delta)$. Then

$$d(x', y') \leq d(x', y) + d(y, y') < \varepsilon + 2\delta = r,$$

i.e. $B_>(y, \varepsilon + \delta) \times B(y, \delta) \subseteq d|_{X\times Y}^{-1}(-\infty, r)$. Moreover, $d(x, y) = \varepsilon < \varepsilon + \delta$ and thus $(x, y) \in B_>(y, \varepsilon + \delta) \times B(y, \delta)$. Thus, $d|_{X\times Y}$ is continuous and hence continuity of $f$ implies continuity of $g$.

Now, let $g$ be continuous and let $z \in Z$. Moreover, let $y \in Y$ and $\varepsilon > 0$ such that $f(z) \in B_>(y, \varepsilon)$. We have to show that there exists an open neighbourhood $V \subseteq Z$ of $z$ such that $f(V) \subseteq B_>(y, \varepsilon)$. Since $g$ is continuous and $g(z, y) = d(f(z), y) < \varepsilon$, there exists some neighbourhood $U \subseteq Z \times Y$ of $(z, y)$ such that $g(U) \subseteq (-\infty, \varepsilon)$. Then $V := \{z' \in Z : (z', y) \in U\}$ is an open subset of $Z$ with $f(V) \subseteq \{x \in X : d(x, y) < \varepsilon\} = B_>(y, \varepsilon)$. Hence, $f$ is continuous. □

Using this lemma we can prove our following main result on admissibility.

**Theorem 3.3.2 (Admissibility)** *Let $(X, Y, d, \alpha)$ be a semi-recursive quasi-metric space. Then the upper Dedekind representation $\delta_{X_>}$ is an admissible representation of $X$ with respect to the weak upper topology.*

**Proof.** First we prove that $\delta_{X_>}$ is continuous with respect to the weak upper topology. By (3) of Definition 3.2.2 $d|_{X\times Y} : X \times Y \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–

computable. Thus,

$$g := d|_{X\times Y} \circ (\mathrm{Inf} \times \mathrm{id}) :\subseteq Y^{\mathbb{N}} \times Y \to \mathbb{R}_>, ((y_n)_{n\in\mathbb{N}}, y) \mapsto d\left(\inf_{n\in\mathbb{N}} y_n, y\right)$$

is $([\delta_Y^\infty, \delta_Y], \delta_{\mathbb{R}_>})$–computable, since $\delta_{X_>} = \mathrm{Inf} \circ \delta_Y^\infty$. Since $\delta_Y^\infty, \delta_Y$ and $\delta_{\mathbb{R}_>}$ are admissible, it follows that $g$ is continuous, thus by Lemma 3.3.1 $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X$ is continuous. Therefore, $\delta_{X_>} = \mathrm{Inf} \circ \delta_Y^\infty$ is continuous.

Now we prove that $\delta_{X_>}$ is topologically maximal among all continuous representations of $X$. Therefore, let $\delta'$ be a continuous representation of $X$ with respect to the weak upper topology. Then $d|_{X\times Y} \circ [\delta', \delta_Y]$ is continuous because of continuity of $\delta_Y$ and Lemma 3.3.1 and thus $d|_{X\times Y}$ is $([\delta', \delta_Y], \delta_{\mathbb{R}_>})$–continuous because of admissibility of $\delta_{\mathbb{R}_>}$. Hence, there is a $q$ such that

$$\delta_{\mathbb{R}_>} \circ \eta_q^{\omega\omega}(p) = d|_{X\times Y} \circ [\delta', \delta_Y](p)$$

for all $p \in \mathrm{dom}[\delta', \delta_Y]$, where $\eta^{\omega\omega}$ is some standard representation of the continuous functions $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $G_\delta$–domain (cf. [Wei00]). Now we define another representation $\delta$ of $X$ by

$$\delta\langle p, r\rangle = \delta'(p) :\Longleftrightarrow r = q$$

for all $p, r \in \mathbb{N}^{\mathbb{N}}$. Using the utm-Theorem for $\eta^{\omega\omega}$ (cf. [Wei00]), it follows that $d|_{X\times Y}$ is $([\delta, \delta_Y], \delta_{\mathbb{R}_>})$–computable. By (4) of Definition 3.2.2 $\delta_{X_>}$ is computably maximal among all such $\delta$, i.e. $\delta \le \delta_{X_>}$. It follows $\delta' \le_{\mathrm{t}} \delta \le \delta_{X_>}$. □

From the proof of the previous theorem we can extract an easy corollary: the infimum operation Inf is continuous with respect to the weak upper topology. The statement is also a direct corollary of the previous theorem and Proposition 3.2.6.

**Corollary 3.3.3** *If $(X, Y, d, \alpha)$ is a semi-recursive quasi-metric space, then* $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X$ *is continuous with respect to the weak upper topology on $X$ and the metric product topology on $Y^{\mathbb{N}}$.*

Together with Proposition 2.3.4 we can conclude that semi-recursive quasi-metric spaces are necessarily continuous from above.

## 3.4 Computability of quasi-metrics

In this section we want to provide some conditions which help to prove that a given quasi-metric space is semi-recursive. As we have seen in the previous section, continuity of Inf is a necessary property of semi-recursive quasi-metric spaces and thus these spaces are necessarily continuous from above. In the following we will prove a sufficient condition which guarantees semi-recursiveness of quasi-metric spaces. In particular, we will use the following prefix condition for quasi-metric spaces.

**Definition 3.4.1 (Prefix-stable quasi-metric spaces)** An upper generated quasi-metric space $(X, Y, d)$ is called *prefix-stable from above*, if $\inf\{y_0, ..., y_k\}$ exists for all $k \in \mathbb{N}$, whenever $(y_n)_{n\in\mathbb{N}}$ is a sequence in $Y$ such that $\inf_{n\in\mathbb{N}} y_n$ exists. Analogously, *prefix-stable from below* can be defined for lower generated quasi-metric spaces.

The following result will be helpful to prove semi-recursiveness of quasi-metric spaces.

**Proposition 3.4.2 (Continuity condition)** *Let $(X, Y, d, \alpha)$ fulfill conditions (1) and (2) of Definition 3.2.2 for semi-recursive quasi-metric spaces. If, additionally,*

*(1) $(X, Y, d)$ is continuous from above and prefix-stable from above,*

*(2) the function $F :\subseteq Y^{\mathbb{N}} \times Y \times \mathbb{N} \to \mathbb{R}_>, ((y_n)_{n\in\mathbb{N}}, y, k) \mapsto d(\inf\{y_0, ..., y_k\}, y)$ is $([\delta_Y^\infty, \delta_Y, \delta_{\mathbb{N}}], \delta_{\mathbb{R}_>})$–computable,*

*then $d : X \times Y \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–computable, i.e. $(X, Y, d, \alpha)$ also fulfills condition (3) of Definition 3.2.2.*

**Proof.** By Proposition 3.2.6 $\mathrm{Inf}_> :\subseteq \mathbb{R}_>^{\mathbb{N}} \to \mathbb{R}_>$ is $(\delta_{\mathbb{R}_>}^\infty, \delta_{\mathbb{R}_>})$–computable. By type conversion (cf. [Wei00]) one can prove that $[F] :\subseteq Y^{\mathbb{N}} \times Y \to \mathbb{R}_>^{\mathbb{N}}$ with $[F]((y_n)_{n\in\mathbb{N}}, y)(n) := F((y_n)_{n\in\mathbb{N}}, y, n)$ is $([\delta_Y^\infty, \delta_Y], \delta_{\mathbb{R}_>}^\infty)$–computable. We recall that $\delta_{X_>} = \mathrm{Inf} \circ \delta_Y^\infty$. Thus, it suffices to prove

$$d \circ (\mathrm{Inf} \times \mathrm{id})((y_n)_{n\in\mathbb{N}}, y) = \mathrm{Inf}_> \circ [F]((y_n)_{n\in\mathbb{N}}, y).$$

for all sequences $(y_n)_{n\in\mathbb{N}} \in \mathrm{dom}(\mathrm{Inf})$ and $y \in Y$. Therefore, let $(y_n)_{n\in\mathbb{N}} \in \mathrm{dom}(\mathrm{Inf})$, $y \in Y$. By assumption $z_k := \inf\{y_0, ..., y_k\}$ exists for all $k \in \mathbb{N}$, $(z_n)_{n\in\mathbb{N}}$ is a decreasing chain, i.e. $z_i \sqsupseteq z_{i+1}$ for all $i \in \mathbb{N}$ and $\inf_{k\in\mathbb{N}} z_k = \inf_{n\in\mathbb{N}} y_n$ exists. Since $d$ is continuous from above, the existence of $\inf_{k\in\mathbb{N}} z_k$ implies

$$\inf_{k\in\mathbb{N}} d(z_k, y) = d\left(\inf_{k\in\mathbb{N}} z_k, y\right) = d\left(\inf_{k\in\mathbb{N}} y_k, y\right),$$

which proves the claim. □

In some cases, where the quasi-metric is computable on the generating space $Y$ and $(Y, \sqsubseteq)$ is an effective semi-lattice (which is the crucial condition) we can slightly simplify the condition of the previous lemma.

**Corollary 3.4.3 (Semi-lattice condition)** *Let $(X, Y, d, \alpha)$ fulfill conditions (1) and (2) of Definition 3.2.2 for semi-recursive quasi-metric spaces. If, additionally,*

*(1) $(X, Y, d)$ is continuous from above,*

*(2) $d|_{Y\times Y} : Y \times Y \to \mathbb{R}$ is $([\delta_Y, \delta_Y], \delta_{\mathbb{R}})$–computable,*

*(3) $\sqcap : Y \times Y \to Y, (y, y') \mapsto \inf\{y, y'\}$ is total and $([\delta_Y, \delta_Y], \delta_Y)$–computable,*

*then $d : X \times Y \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–computable, i.e. $(X, Y, d, \alpha)$ also fulfills condition (3) of Definition 3.2.2.*

## 3.5 Metric and quasi-metric spaces

If $(X, X, d, \alpha)$ is a recursive quasi-metric space, i.e. if we are in the situation that the generating subspace is the whole space, then we canonically have three representations of $X$, the Cauchy representation $\delta_X$ of $(X, d_\star, \alpha)$, the lower Dedekind representation $\delta_{X_<} := \mathrm{Sup} \circ \delta_X^\infty$ and the upper Dedekind representation $\delta_{X_>} := \mathrm{Inf} \circ \delta_X^\infty$. The question arises, how these representations are related. On the one hand, it is clear that $\delta_X \leq \delta_{X_<}$ and $\delta_X \leq \delta_{X_>}$, since the corresponding injections are computable. On the other hand, it is not clear whether $\delta_X$ is an infimum of $\delta_{X_<}$ and $\delta_{X_>}$ in the lattice of representations. An infimum of two representations $\delta, \delta'$ of a set $X$ is given by their *conjunction* $\delta \sqcap \delta'$, which is a representation of $X$, defined by

$$(\delta \sqcap \delta')\langle p, q\rangle = x :\iff \delta(p) = x \text{ and } \delta'(q) = x$$

for all $p, q$ and $x \in X$. It is easy to see that $\delta \sqcap \delta'$ is an infimum of $\delta$ and $\delta'$ in the lattice of representations (cf. [Wei87]). Hence, we can reformulate our question as follows: under which conditions does $\delta_X \equiv \delta_{X_<} \sqcap \delta_{X_>}$ hold? The following proposition formulates sufficient conditions.

**Theorem 3.5.1 (Conjunction of quasi-metric spaces)** *Let $(X, X, d, \alpha)$ be a recursive quasi-metric space such that*

*(1) $(X, X, d)$ is prefix-stable from below,*

*(2) $\sqcup :\subseteq X \times X \to X, (x, x') \mapsto \sup\{x, x'\}$ is $([\delta_X, \delta_X], \delta_X)$–computable.*

*Then $\delta_X \equiv \delta_{X_<} \sqcap \delta_{X_>}$.*

**Proof.** Since the function $X \to X^{\mathbb{N}}$, which maps each point $x \in X$ to the constant sequence with value $x$, is computable with respect to $\delta_X$, we obtain $\delta_X \leq \delta_{X_<} \sqcap \delta_{X_>}$.

It remains to show $\delta_{X_<} \sqcap \delta_{X_>} \leq \delta_X$. Let sequences $\langle p_i \rangle_{i\in\mathbb{N}}$ and $q$ with $x := \delta_{X_<} \sqcap \delta_{X_>}\langle\langle p_i\rangle_{i\in\mathbb{N}}, q\rangle$, $y_i := \delta_X(p_i)$ be given. Thus, $\sup_{n\in\mathbb{N}} y_n = x$ and since $(X, X, d)$ is prefix-stable from below $z_n := \sup\{y_0, ...y_n\}$ exists for all $n \in \mathbb{N}$ and hence $\sup_{n\in\mathbb{N}} z_n = x$. By Proposition 3.2.4 and Proposition 2.3.4 it follows that $(X, X, d)$ is consistent from below with respect to the lower topology. Thus, the sequence $(z_n)_{n\in\mathbb{N}}$ converges to $x$ with respect to the lower topology, i.e. for each $k \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $z_n \in B_<(x, 2^{-k-1})$. Thus, for each $k \in \mathbb{N}$ we can effectively find some $n \in \mathbb{N}$ such that

$$d(x, y_0 \sqcup ... \sqcup y_n) < 2^{-k-1},$$

since $d : X \times X \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_X], \delta_{\mathbb{R}_>})$–computable and $\sqcup :\subseteq X \times X \to X$ is $([\delta_X, \delta_X], \delta_X)$–computable. Thus, we can find an $r_k$ such that $\delta_X(r_k) = y := y_0 \sqcup ... \sqcup y_n$. Since $y \sqsubseteq x$, we obtain $d(y, x) = 0$ and thus $d_\star(x, y) < 2^{-k-1}$. Since the limit operation $\text{Lim} :\subseteq X^{\mathbb{N}} \to X$ is computable, we can effectively find some $r$ such that $\delta_X(r) = \lim_{k\to\infty} \delta_X(r_k)$, i.e. $\delta_X(r) = x = \delta_{X_<} \sqcap \delta_{X_>}\langle\langle p_i\rangle_{i\in\mathbb{N}}, q\rangle$. □

By symmetry, an analogous property holds if $(X, X, d)$ is prefix-stable from above and $\sqcap$ is computable. Under the same assumptions one could also prove that $d : X_> \times X_< \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_{X_<}], \delta_{\mathbb{R}_>})$–computable.

## 3.6 Effective generation of quasi-metric spaces

In this section we want to discuss sufficient additional conditions which guarantee that condition (4) in Definition 3.2.2 is a direct consequence of conditions (1) to (3). These purely topological conditions will be applied in succeeding sections to concrete quasi-metric spaces.

Essentially, we will prove an effective version of Proposition 2.4.3. We show that a space which is upper generated and fulfills some of the conditions of a semi-recursive quasi-metric space, is also effectively upper generated. The essential idea is the same as in the proof of Proposition 2.4.3, although the effectivization needs some care.

**Proposition 3.6.1 (Effective upper generation)** *Let $(X, Y, d, \alpha)$ fulfill conditions (1) to (3) of Definition 3.2.2 for semi-recursive quasi-metric spaces, let $Y$ be strongly dense in $(X, d)$, and let $(Y, d_\star)$ be complete. Then $(X, Y, d, \alpha)$ also fulfills condition (4) and thus it is a semi-recursive quasi-metric space.*

**Proof.** Let $Y \subseteq X$ be strongly dense in $(X, d)$ with constant $c \geq 1$. Let us assume that $\delta$ is a representation of $X$ such that $d : X \times Y \to \mathbb{R}$ is $([\delta, \delta_Y], \delta_{\mathbb{R}_>})$–computable. We have to prove $\delta \leq \delta_{X_>}$. Given a name $p$ of $x$, i.e. $\delta(p) = x$, we can find values $l \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that $d(x, \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$ and given these, we can effectively find a sequence $(m_k)_{k \in \mathbb{N}}$ such that

$$d_\star(\alpha(m_{k+1}), \alpha(m_k)) < 2^{-k-l-1} \text{ and } d(x, \alpha(m_k)) < \frac{1}{c} \cdot 2^{-k-l-2} \quad (*)$$

for all $k \in \mathbb{N}$, since $(Y, d_\star, \alpha)$ is a recursive metric space and $d$ is $([\delta, \delta_Y], \delta_{\mathbb{R}_>})$–computable. By induction one can prove that such a sequence exists since $Y \subseteq X$ is strongly dense with constant $c$ and $\alpha$ is dense in $(Y, d_\star)$. Moreover, $(\alpha(m_k))_{k \in \mathbb{N}}$ is a Cauchy sequence and since $(Y, d_\star)$ is complete, the limit $y := \lim_{k \to \infty} \alpha(m_k)$ exists in $Y$. We obtain

$$d(x, y) \leq d(x, \alpha(m_k)) + d_\star(\alpha(m_k), y) < \frac{1}{c} \cdot 2^{-k-l-2} + 2^{-k-l} < 2^{-k-l+1}$$

for all $k \in \mathbb{N}$ and hence $d(x, y) = 0$ and $x \sqsubseteq y$. For any $l$ and any starting point $m_0$ with $d(x, \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$ we have found a point $y$. Let $V$ be the set of all these points (for arbitrary suitable $m_0 \in \mathbb{N}$ and $l \in \mathbb{N}$). Since our procedure was effective in $p$, we can effectively find a sequence $q = \langle q_0, q_1, ... \rangle$ which enumerates $V$, i.e. $V = \{\delta_Y(q_i) : i \in \mathbb{N}\}$. Since $(X, Y, d)$ is upper generated, there is some set $W \subseteq \{y \in Y : x \sqsubseteq y\} =: U$ such that $x = \inf W$. By Lemma 2.4.2 it suffices to prove that $V$ is dense in $U$ because this implies $\delta(p) = x = \inf V = \mathrm{Inf} \circ \delta_Y^\infty(q) = \delta_{X_>}(q)$. Therefore, let $y' \in U$ and $\varepsilon > 0$. Then there is some $l \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that $d_\star(y', \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$ and $2^{-l+1} < \varepsilon$. Hence $d(x, \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$ and there is a corresponding sequence $(m_k)_{k \in \mathbb{N}}$ which fulfills $(*)$ and $y := \lim_{k \to \infty} \alpha(m_k) \in V$. We obtain

$$d_\star(y', y) \leq d_\star(y', \alpha(m_0)) + d_\star(\alpha(m_0), y) < \frac{1}{c} \cdot 2^{-l-2} + 2^{-l} < 2^{-l+1} < \varepsilon.$$

This completes the proof. □

One should mention that we have not used condition (3) of Definition 3.2.2 for the proof. The properties given in the previous proposition lead to the following formally stronger alternative definition of recursive quasi-metric spaces. Here, Property (4) of Definition 3.2.2 is replaced by purely topological conditions.

**Definition 3.6.2 (Strong semi-recursive quasi-metric spaces)** We call $(X, Y, d, \alpha)$ a *strong semi-recursive quasi-metric space*, if the following applies:

(1) $(X, Y, d)$ is an upper generated quasi-metric space with upper Dedekind representation $\delta_{X_>}$, such that $Y$ is strongly dense in this space,

(2) $(Y, d_\star|_{Y\times Y}, \alpha)$ is a complete recursive metric space with Cauchy representation $\delta_Y$,

(3) $d|_{X\times Y} : X \times Y \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_Y], \delta_{\mathbb{R}_>})$–computable,

We will call $(X, Y, d, \alpha)$ a *strong recursive quasi-metric space*, if $(X, Y, d, \alpha)$ as well as the conjugate space $(X, Y, \overline{d}, \alpha)$ are strong semi-recursive quasi-metric spaces.

All concrete examples of semi-recursive quasi-metric spaces which we will define, fulfill the previous stronger definition. Now Proposition 3.6.1 can be reformulated as follows.

**Corollary 3.6.3** *Any strong semi-recursive quasi-metric space is a semi-recursive quasi-metric space.*

## 3.7 The hyperspace of compact subsets

In this section we want to show that the set $\mathcal{K}(X)$ of non-empty compact subsets of a metric space $(X, d)$ can be endowed with the structure of a semi-recursive quasi-metric space. If $(X, d, \alpha)$ is a separable metric space, then $(\mathcal{K}(X), d_\mathcal{K}, \alpha_\mathcal{K})$ with the Hausdorff metric $d_\mathcal{K}$ and a sequence $\alpha_\mathcal{K} : \mathbb{N} \to \mathcal{K}(X)$ with the set $\mathcal{F}(Q)$ of non-empty finite subsets of $Q := \text{range}(\alpha)$ as range, defined by

$$\alpha_\mathcal{K}\langle\langle n_0, ..., n_k\rangle, k\rangle := \{\alpha(n_0), ..., \alpha(n_k)\},$$

is a separable metric space too. It is well-known, that completeness of $(X, d)$ implies completeness of $(\mathcal{K}(X), d_\mathcal{K})$. Similar as for the Euclidean case we obtain (cf. Proposition 4.4.32 in [Bra99a]):

**Lemma 3.7.1** *If $(X, d, \alpha)$ is a (complete) recursive metric space, then also $(\mathcal{K}(X), d_\mathcal{K}, \alpha_\mathcal{K})$ is a (complete) recursive metric space.*

Moreover, it is straightforward to prove that the quasi-metric $d'_\mathcal{K}$ is computable with respect to the Cauchy representation $\delta_\mathcal{K}$ of $(\mathcal{K}(X), d_\mathcal{K}, \alpha_\mathcal{K})$ (cf. Proposition 4.4.35 and Corollary 4.4.40 in [Bra99a]).

**Lemma 3.7.2** *The quasi-metric $d'_\mathcal{K} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}$ is $([\delta_\mathcal{K}, \delta_\mathcal{K}], \delta_\mathbb{R})$–computable.*

Now we are prepared to prove that under suitable assumptions $(\mathcal{K}, \mathcal{K}, d'_\mathcal{K}, \alpha_\mathcal{K})$ and its conjugate space are strong semi-recursive quasi-metric spaces.

**Proposition 3.7.3 (Space of compact subsets)** *If $(X, d, \alpha)$ is a complete recursive metric space, then $(\mathcal{K}(X), \mathcal{K}(X), d'_\mathcal{K}, \alpha_\mathcal{K})$ is a strong semi-recursive quasi-metric space.*

**Proof.** We have to verify conditions (1) to (3) of Definition 3.6.2.

(1) This has been proved in Theorem 2.5.2.

(2) This has been stated in Lemma 3.7.1.

(3) We have to show that $d'_\mathcal{K}$ is $([\delta_{\mathcal{K}_>}, \delta_\mathcal{K}], \delta_{\mathbb{R}_>})$–computable. Since $(\mathcal{K}, \mathcal{K}, d'_\mathcal{K})$ is obviously prefix-stable from above and continuous from above by Theorem 2.5.2, it suffices to prove by Proposition 3.4.2 that

$$F :\subseteq \mathcal{K}^\mathbb{N} \times \mathcal{K} \times \mathbb{N} \to \mathbb{R}_>, ((A_n)_{n\in\mathbb{N}}, B, m) \mapsto d'_\mathcal{K}\left(\bigcap_{n=0}^{m} A_n, B\right)$$

is $([\delta_\mathcal{K}^\infty, \delta_\mathcal{K}, \delta_\mathbb{N}], \delta_{\mathbb{R}_>})$–computable. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{K}^\mathbb{N}$, $B \in \mathcal{K}$, and $m \in \mathbb{N}$ such that $A := \bigcap_{n=0}^m A_n \neq \emptyset$ and let $S_{nk}, T_k$ be finite subsets of $Q := \mathrm{range}(\alpha)$, such that $d_\mathcal{K}(A_n, S_{nk}) < 2^{-k}$ and $d_\mathcal{K}(B, T_k) < 2^{-k}$ for all $n, k \in \mathbb{N}$. We define sets

$$D_{kl} := \{r \in S_{0l} : (\forall n = 0, ..., m)(\exists s \in S_{nl})\ d(r, s) < 2^{-k}\}$$

for all $k \in \mathbb{Z}$, $l \in \mathbb{N}$. Then we can effectively find some set $R_k$ with $D_{k-1k} \subseteq R_k \subseteq D_{k-2k}$ for all $k \in \mathbb{N}$. By Lemma 3.7.2 it suffices to prove

$$d'_\mathcal{K}(A, B) = \inf_{k\in\mathbb{N}} (d'_\mathcal{K}(R_k, T_k) + 2^{-k+1}) \quad (*)$$

in order to obtain computability of $F$. If we can show

(a) $(\forall k)\ R_k \neq \emptyset$ and $d'_\mathcal{K}(A, R_k) \leq 2^{-k}$,

(b) $(\forall \varepsilon > 0)(\forall l)(\exists k > l)\ d'_\mathcal{K}(R_k, A) < \varepsilon$,

then it follows

$$d'_\mathcal{K}(A, B) \leq d'_\mathcal{K}(A, R_k) + d'_\mathcal{K}(R_k, T_k) + d_\mathcal{K}(T_k, B) < d'_\mathcal{K}(R_k, T_k) + 2^{-k+1}$$

for all $k \in \mathbb{N}$; for each $\delta > 0$ there is an $\varepsilon > 0$ and a $k \in \mathbb{N}$ such that $\varepsilon + 3 \cdot 2^{-k} < \delta$ and $d'_\mathcal{K}(R_k, A) < \varepsilon$ and thus

$$\begin{aligned} d'_\mathcal{K}(R_k, T_k) + 2^{-k+1} &\leq d'_\mathcal{K}(R_k, A) + d'_\mathcal{K}(A, B) + d_\mathcal{K}(B, T_k) + 2^{-k+1} \\ &< \varepsilon + d'_\mathcal{K}(A, B) + 3 \cdot 2^{-k} \\ &< d'_\mathcal{K}(A, B) + \delta. \end{aligned}$$

Together, this proves Equation $(*)$. It remains to prove (a) and (b).

(a) Let $k \in \mathbb{N}$ and $x \in A = \bigcap_{n=0}^{m} A_n$. Then for all $n = 0, ..., m$ there is some $s_n \in S_{nk}$ such that $d(x, s_n) < 2^{-k}$. With $r := s_0$ we obtain $d(r, s_n) < 2^{-k+1}$ for all $n = 0, ..., m$ and thus $r \in D_{k-1k} \subseteq R_k$. Consequently, $d'_{\mathcal{K}}(A, R_k) = \sup_{x\in A} d_{R_k}(x) \le 2^{-k}$ for all $k \in \mathbb{N}$.

(b) Let $\varepsilon > 0$ and $l \in \mathbb{N}$. Since $R_k$ is finite there is some $r_k \in R_k$ for each $k \in \mathbb{N}$ such that $d'_{\mathcal{K}}(R_k, A) = \sup_{r\in R_k} d_A(r) = d_A(r_k)$. Since $R_k \subseteq D_{k-2k}$ there is some $s_{nk} \in S_{nk}$ for each $n = 0, ..., m$ such that $d(r_k, s_{nk}) < 2^{-k+2}$ and some $x_{nk} \in A_n$ such that $d(s_{nk}, x_{nk}) < 2^{-k}$ for each $k \in \mathbb{N}$. Since $A_0$ is compact, $(x_{0k})_{k\in\mathbb{N}}$ has a convergent subsequence $(x'_{0k})_{k\in\mathbb{N}}$. Let $(x'_{nk})_{k\in\mathbb{N}}$ for all $n = 0, ..., m$ and $(r'_k)_{k\in\mathbb{N}}$ be the corresponding subsequence of $(x_{nk})_{k\in\mathbb{N}}$ and $(r_k)_{k\in\mathbb{N}}$, respectively. Since $d(r_k, x_{nk}) < 2^{-k+3}$ for all $n = 0, ..., m$ we obtain $\lim_{k\to\infty} x'_{0k} = \lim_{k\to\infty} x'_{nk} = \lim_{k\to\infty} r'_k =: x$ for all $n = 0, ..., m$ and $x \in \bigcap_{n=0}^{m} A_n = A$ since $A_0, ..., A_m$ are complete. Now there is some $k' > l$ such that $d(x, r'_{k'}) < \varepsilon$ and there is some $k > l$ such that $r_k = r'_{k'}$. Thus, $d'_{\mathcal{K}}(R_k, A) = d_A(r_k) \le d(x, r'_{k'}) < \varepsilon$.

□

Now we investigate the conjugate quasi-metric space.

**Proposition 3.7.4 (Conjugate space of compact subsets)** *If $(X, d, \alpha)$ is a complete recursive metric space, then $(\mathcal{K}(X), \mathcal{K}(X), \overline{d'_{\mathcal{K}}}, \alpha_{\mathcal{K}})$ is a strong semi-recursive quasi-metric space.*

**Proof.** We have to verify conditions (1) to (3) of Definition 3.6.2.

(1) This has been proved in Theorem 2.5.2.

(2) This has been stated in Lemma 3.7.1.

(3) We have to show that $\overline{d'_{\mathcal{K}}}$ is $([\delta_{\mathcal{K}_<}, \delta_{\mathcal{K}}], \delta_{\mathbb{R}_>})$–computable. We will apply Corollary 3.4.3. First we note that $\sqcup : \mathcal{K} \times \mathcal{K} \to \mathcal{K}, (A, B) \mapsto A \cup B$ is $([\delta_{\mathcal{K}}, \delta_{\mathcal{K}}], \delta_{\mathcal{K}})$–computable (cf. Proposition 4.4.34 in [Bra99a]). By Theorem 2.5.2 $(\mathcal{K}, \mathcal{K}, \overline{d'_{\mathcal{K}}})$ is continuous from above and by Lemma 3.7.2 $\overline{d'_{\mathcal{K}}} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is $([\delta_{\mathcal{K}}, \delta_{\mathcal{K}}], \delta_{\mathbb{R}})$–computable. □

Altogether we obtain the following result. By Theorem 3.5.1 we can conclude that under the same assumptions $\delta_{\mathcal{K}} \equiv \delta_{\mathcal{K}_<} \sqcap \delta_{\mathcal{K}_>}$ holds for the corresponding representations.

**Corollary 3.7.5 (Space of compact subsets)** *If $(X, d, \alpha)$ is a complete recursive metric space, then $(\mathcal{K}(X), \mathcal{K}(X), d'_{\mathcal{K}}, \alpha_{\mathcal{K}})$ is a strong recursive quasi-metric space.*

## 3.8 The spaces of semi-continuous functions

In this section we want to endow the sets $\mathcal{LSC}(X)$, $\mathcal{USC}(X)$ of lower and upper semi-continuous functions $f : X \to \mathbb{R}$, defined on a metric space $X$, with the structure of a semi-recursive quasi-metric space. In this section we will use the notion of a recursively locally compact metric space. We recall that a Hausdorff space is called *locally compact*, if each point has a compact neighbourhood. In Lemma 2.6.1 we have shown that a separable metric space is locally compact, if and only if it admits an exhausting sequence, i.e. if it can be represented as a countable union of compact subsets in a certain way. We use this characterization for the definition of recursively locally compact separable metric spaces.

**Definition 3.8.1 (Recursively locally compact metric spaces)** A separable metric space $X$ is called *recursively locally compact*, if there is a sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets $K_i \in \mathcal{K}(X)$ such that:

(1) $X = \bigcup_{i=0}^{\infty} K_i$ and $K_i \subseteq K_{i+1}^{\circ}$ for all $i \in \mathbb{N}$,

(2) $(K_i)_{i\in\mathbb{N}}$ is a computable sequence in $\mathcal{K}(X)$,

(3) $(K_i^{\circ})_{i\in\mathbb{N}}$ is a recursively given sequence of r.e. open sets.

In this situation $(K_i)_{i\in\mathbb{N}}$ is called a *recursive exhausting sequence* of $X$.

A sequence $(A_n)_{n\in\mathbb{N}}$ of sets is called a *recursively given sequence of r.e. open sets*, if there is a computable function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that

$$A_i = \bigcup\{B(\alpha(n), 2^{-k}) : (\exists j) f(j,i) = (n,k)\},$$

for all $i \in \mathbb{N}$ (cf. [Bra99a]) and a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(X)$ is called *computable*, if and only if it is a computable sequence in the computable metric space $(\mathcal{K}(X), d_{\mathcal{K}}, \alpha_{\mathcal{K}})$ (i.e. if it is $\delta_{\mathcal{K}}^{\infty}$–computable).

It is easy to see that $(\mathcal{C}(X), d_{\mathcal{C}})$ is a complete metric space. If $D \subseteq X$ is a dense subset of $X$, then the set $\mathcal{F} := \{d_q : q \in D\} \subseteq \mathcal{C}(X)$, with the distance functions $d_x : X \to \mathbb{R}, y \mapsto d(x,y)$, separates points and thus by the Stone-Weierstraß Approximation Theorem the set $\mathcal{P}$ of polynomials in functions of $\mathcal{F}$ with coefficients from $\mathbb{Q}$ is dense in $\mathcal{C}(X)$ (cf. [Bou66a]). If $(X, d, \alpha)$ is a separable metric space, then we can define a canonical numbering $\alpha_{\mathcal{C}(X)}$ of the set $\mathcal{P}$ by

$$\alpha_{\mathcal{C}(X)}\langle\langle l_{01},...,l_{0j_0},k_0,j_0\rangle,...,\langle l_{n1},...,l_{nj_n},k_n,j_n\rangle,n\rangle := \sum_{i=0}^{n}\alpha_{\mathbb{R}}(k_i)\prod_{j=1}^{j_i} d_{\alpha(l_{ij})}.$$

With this definition, $(\mathcal{C}(X), d_{\mathcal{C}}, \alpha_{\mathcal{C}(X)})$ is a separable metric space. Finally, this space is recursive, if $(X, d, \alpha)$ is a recursively locally compact recursive metric space (cf. Proposition 4.4.53 in [Bra99a]). It follows that $(d'_{\mathcal{C}})_\star$ is computable too and thus by Inequality (2.1) it follows that $(d'_{\mathcal{C}})_\star$ is recursively related to $d_{\mathcal{C}}$, which roughly speaking means that we do not have to distinguish both metrics from point of view of computability (cf. Proposition 4.4.22 in [Bra99a]). Our goal is to prove the following theorem which shows that a similar property holds for the space of semi-continuous functions.

**Theorem 3.8.2 (Space of semi-continuous functions)** *If $(X, d, \alpha)$ is a recursively locally compact and recursive metric space, then the quadruple $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}}, \alpha_{\mathcal{C}(X)})$ is a strong semi-recursive quasi-metric space. The same holds for $(\mathcal{LSC}(X), \mathcal{C}(X),\ d_{\mathcal{LSC}},\ \alpha_{\mathcal{C}(X)})$.*

**Proof.** We consider the case $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}}, \alpha_{\mathcal{C}(X)})$. The space $\mathcal{LSC}(X)$ can be treated correspondingly.

(1) By Theorem 2.6.4, $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is an upper generated quasi-metric space and $\mathcal{C}(X)$ is strongly dense in this space.

(2) Moreover, $(\mathcal{C}(X), d_{\mathcal{C}}, \alpha_{\mathcal{C}(X)})$ is a complete recursive metric space (cf. Proposition 4.4.53 in [Bra99a]).

(3) We have to show that $d_{\mathcal{USC}}|_{\mathcal{USC}(X)\times\mathcal{C}(X)} : \mathcal{USC}(X) \times \mathcal{C}(X) \to \mathbb{R}_>$ is $([\delta_{\mathcal{USC}(X)_>}, \delta_{\mathcal{C}(X)}],\ \delta_{\mathbb{R}_>})$–computable. Therefore, we will apply Corollary 3.4.3. By Theorem 2.6.4 $(\mathcal{USC}(X), \mathcal{C}(X), d_{\mathcal{USC}})$ is continuous from above. Next, we note that using some well-known techniques (evaluation and type-conversion, cf. Theorem 4.4.55 in [Bra99a]) one can prove that the infimum map $\sqcap : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X), (f, g) \mapsto \min(f, g)$ is $([\delta_{\mathcal{C}(X)}, \delta_{\mathcal{C}(X)}], \delta_{\mathcal{C}(X)})$–computable, since $\min : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $([\delta_{\mathbb{R}}, \delta_{\mathbb{R}}], \delta_{\mathbb{R}})$–computable. Moreover, using some known facts it is easy to prove that $d_{\mathcal{USC}}|_{\mathcal{C}(X)\times\mathcal{C}(X)} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}$ is $([\delta_{\mathcal{C}(X)}, \delta_{\mathcal{C}(X)}], \delta_{\mathbb{R}})$–computable (cf. the proof of Proposition 4.4.53 in [Bra99a]). Therefore, one can use the fact that the functions $\sigma : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}, (x_n)_{n\in\mathbb{N}} \mapsto \sum_{i=0}^{\infty} 2^{-i-1}\frac{x_i}{1+x_i}$, $\dot{-} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sup : \mathcal{K}(\mathbb{R}) \to \mathbb{R}$ and the compact image mapping $\mathcal{C}(X) \times \mathcal{K}(X) \to \mathcal{K}(X), (f, K) \mapsto f(K)$ are computable with respect to corresponding representations.

$\square$

It is easy to see that $(\mathbb{R}^n, d_{\mathbb{R}^n}, \alpha_{\mathbb{R}^n})$ can be considered as a (nice) recursively locally compact recursive metric space with respect to the exhausting sequence $([-i, i]^n)_{i\in\mathbb{N}}$ (cf. Example 4.4.46 in [Bra99a]). Thus, by the previous theorem $(\mathcal{USC}(\mathbb{R}^n), \mathcal{C}(\mathbb{R}^n), d_{\mathcal{USC}}, \alpha_{\mathcal{C}(\mathbb{R}^n)})$ is a semi-recursive quasi-metric space. The final proposition in this section states that the computable points of this space are well-known (cf. [WZ00, ZBW99, Wei00]). We write for short $\delta_{\mathcal{USC}(X)}$ instead of $\delta_{\mathcal{USC}(X)_>}$ and $\delta_{\mathcal{LSC}(X)}$ instead of $\delta_{\mathcal{LSC}(X)_>}$.

**Proposition 3.8.3** *The $\delta_{\mathcal{USC}(\mathbb{R}^n)}$–computable points of the semi-recursive quasi-metric space $(\mathcal{USC}(\mathbb{R}^n), \mathcal{C}(\mathbb{R}^n), d_{\mathcal{USC}}, \alpha_{\mathcal{C}(\mathbb{R}^n)})$ are exactly the upper semi-computable functions $f : \mathbb{R}^n \to \mathbb{R}$.*

A corresponding property holds in case of $\mathcal{LSC}(\mathbb{R}^n)$. We postpone the proof to Chapter 5 (it is a direct consequence of Theorem 5.1.2).

## 3.9 The hyperspace of closed subsets

In this section we want to endow the set $\mathcal{A}(X)$ of non-empty closed subsets of a locally compact separable metric space $(X, d)$ with the structure of a semi-recursive quasi-metric space. For short, we will sometimes write $\mathcal{A}$ instead of $\mathcal{A}(X)$. If $(X, d, \alpha)$ is a nice locally compact separable metric space, then the space $(\mathcal{A}(X), d_{\mathcal{A}}, \alpha_{\mathcal{A}})$ with some standard numbering $\alpha_{\mathcal{A}}$ of the non-empty finite subsets of range($\alpha$), defined by $\alpha_{\mathcal{A}}(n) := \alpha_{\mathcal{K}}(n)$, is a complete separable metric space too. An analogous property holds with respect to recursiveness (cf. Proposition 4.4.66 in [Bra99a]):

**Lemma 3.9.1** *If $(X, d, \alpha)$ is a nice recursively locally compact recursive metric space, then $(\mathcal{A}(X), d_{\mathcal{A}}, \alpha_{\mathcal{A}})$ is a complete recursive metric space.*

Moreover, it is straightforward to prove that the quasi-metric $d'_{\mathcal{A}}$ is computable with respect to the Cauchy representation $\delta_{\mathcal{A}}$ of $(\mathcal{A}(X), d_{\mathcal{A}}, \alpha_{\mathcal{A}})$.

**Lemma 3.9.2** *If $(X, d)$ is a nice recursively locally compact recursive metric space, then $d'_{\mathcal{A}} : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}$ is $([\delta_{\mathcal{A}}, \delta_{\mathcal{A}}], \delta_{\mathbb{R}})$–computable.*

**Proof.** By Lemma 2.7.6 we obtain

$$d'_{\mathcal{A}}(A, B) = d'_{\mathcal{A}}(A \cup B, B) = d_{\mathcal{A}}(A \cup B, B)$$

for all $A, B \in \mathcal{A}(X)$. Thus, $d'_{\mathcal{A}}$ is $([\delta_{\mathcal{A}}, \delta_{\mathcal{A}}], \delta_{\mathbb{R}})$–computable, since $d_{\mathcal{A}}$ and $\cup$ are computable with respect to $\delta_{\mathcal{A}}$ (cf. Theorem 4.4.67 in [Bra99a]). □

It follows that $(d'_{\mathcal{A}})_\star$ is computable too and thus by Inequality (2.2) it follows that $(d'_{\mathcal{A}})_\star$ is recursively related to $d_{\mathcal{A}}$, which roughly speaking means that we do not have to distinguish both metrics from point of view of computability (cf. Proposition 4.4.22 in [Bra99a]).

Now we will prove that under suitable assumptions $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}}, \alpha_{\mathcal{A}})$ and its conjugate space are semi-recursive quasi-metric spaces. The proof is a refined version of the proof of the corresponding Proposition 3.7.3 for the space of compact subsets.

**Proposition 3.9.3 (Space of closed subsets)** *If $(X, d, \alpha)$ is a nice recursively locally compact recursive metric space, then $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}}, \alpha_{\mathcal{A}})$ is a strong semi-recursive quasi-metric space.*

**Proof.** We have to verify conditions (1) to (3) of Definition 3.6.2.

(1) This has been proved in Theorem 2.7.7.

(2) This has been stated in Lemma 3.9.1.

(3) We have to show that $d'_{\mathcal{A}}$ is $([\delta_{\mathcal{A}_>}, \delta_{\mathcal{A}}], \delta_{\mathbb{R}_>})$–computable. Since $(\mathcal{A}, \mathcal{A}, d'_{\mathcal{A}})$ is obviously prefix-stable from above and continuous from above by Theorem 2.7.7 it suffices to prove by Proposition 3.4.2 that

$$F :\subseteq \mathcal{A}^{\mathbb{N}} \times \mathcal{A} \times \mathbb{N} \to \mathbb{R}_>, ((A_n)_{n\in\mathbb{N}}, B, m) \mapsto d'_{\mathcal{A}}\left(\bigcap_{n=0}^{m} A_n, B\right)$$

is $([\delta_{\mathcal{A}}^{\infty}, \delta_{\mathcal{A}}, \delta_{\mathbb{N}}], \delta_{\mathbb{R}_>})$–computable. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, $B \in \mathcal{A}$, and $m \in \mathbb{N}$ such that $A := \bigcap_{n=0}^{m} A_n \neq \emptyset$ and let $j := \min\{i \in \mathbb{N} : K_i \cap A \neq \emptyset\}$. Let $S_{nk}, T_k, K_{ik}$ be finite subsets of $Q := \mathrm{range}(\alpha)$, such that $d_{\mathcal{A}}(A_n, S_{nk}) < 2^{-k}$, $d_{\mathcal{A}}(B, T_k) < 2^{-k}$ and $d_{\mathcal{K}}(K_i, K_{ik}) < 2^{-k}$ for all $n, k, i \in \mathbb{N}$. We define sets

$$D_{kl} := \left\{ r \in \bigcup_{i=0}^{k} K_{il} : (\forall n = 0, ..., m)\ d'_{\mathcal{A}}(\{r\}, S_{nl}) < 2^{-k} \right\}$$

for all $k \in \mathbb{Z}$, $l \in \mathbb{N}$. Then we can effectively find some set $R_k$ with $D_{k-1k} \subseteq R_k \subseteq D_{k-2k}$ for all $k \in \mathbb{N}$. By Lemma 3.9.2 it suffices to prove

$$d'_{\mathcal{A}}(A, B) = \inf_{\substack{k\in\mathbb{N} \\ R_k \neq \emptyset}} (d'_{\mathcal{A}}(R_k, T_k) + 2^{-k+4}) \quad (*)$$

in order to obtain computability of $F$ (here it should be noticed that $R_k \neq \emptyset$ is an r.e. property in $k$). If we can show

(a) $(\forall k)(\ (k > j \Longrightarrow R_k \neq \emptyset)$ and $(R_k \neq \emptyset \Longrightarrow d'_{\mathcal{A}}(A, R_k) \leq 2^{-k+3})\ )$,

(b) $(\forall \varepsilon > 0)(\forall l)(\exists k > j + l)\ d'_{\mathcal{A}}(R_k, A) < \varepsilon$,

then Equation $(*)$ follows as in the proof of Proposition 3.7.3. It remains to prove (a) and (b).

(a) Let $k > j$. By definition of $j$ there is some $x \in A \cap K_j$. Then there is some $r \in K_{jk}$ such that $d(r,x) < 2^{-k}$. Thus, by Equation (2.3)

$$\begin{aligned} d'_{\mathcal{A}}(\{r\}, S_{nk}) &\leq d'_{\mathcal{A}}(\{r\},\{x\}) + d'_{\mathcal{A}}(\{x\}, S_{nk}) \\ &\leq d(r,x) + d'_{\mathcal{A}}(A_n, S_{nk}) \\ &< 2^{-k} + 2^{-k} = 2^{-k+1} \end{aligned}$$

for all $n = 0, ..., m$, i.e. $r \in D_{k-1k} \subseteq R_k$. Especially, $R_k \neq \emptyset$.
Now let $k \in \mathbb{N}$ with $R_k \neq \emptyset$. For each $i \in \mathbb{N}$ there is some $y_i \in K_i$ such that $|d_{R_k} \dot{-} d_A|_{K_i} = d_{R_k}(y_i) \dot{-} d_A(y_i)$.
Now let us assume that there is some $i < j-1$ with $d_A(y_i) < 1$. Then there is some $x \in A$ with $d(x, y_i) < 1$ and thus $x \in K_{i+1}$ since $d$ is nice. But this implies $K_{i+1} \cap A \neq \emptyset$ and thus $i+1 \geq j$. Contradiction! In other words,

$$|d_{R_k} \dot{-} d_A|_{K_i} = d_{R_k}(y_i) \dot{-} d_A(y_i) = 0$$

for all $i < j-1$.
<u>1. Case: $j \geq k-1$.</u> Then

$$d'_{\mathcal{A}}(A, R_k) \leq 2^{-j+1} + \sum_{i=0}^{j-2} 2^{-i-1} |d_{R_k} \dot{-} d_A|_{K_i} \leq 2^{-j+1} \leq 2^{-k+2}.$$

<u>2. Case: $j < k-1$.</u> Let $i \in \mathbb{N}$ with $j \leq i < k-1$. Since $A \cap K_i \neq \emptyset$, we can assume without loss of generality $d_A(y_i) < 1$. Thus, there is some $x_i \in A$ with $d(x_i, y_i) < 1$ and $d(x_i, y_i) < d_A(y_i) + 2^{-k}$. Since $y_i \in K_i$ and $d$ is nice, we obtain $x_i \in K_{i+1}$. Thus, there is some $r_i \in K_{i+1k}$ with $d(r_i, x_i) < 2^{-k}$. As above, it follows $d'_{\mathcal{A}}(\{r_i\}, S_{nk}) < 2^{-k+1}$ for all $n = 0, ..., m$ and thus $r_i \in D_{k-1k} \subseteq R_k$ since $i+1 \leq k-1$. Moreover

$$\begin{aligned} |d_{R_k} \dot{-} d_A|_{K_i} &= d_{R_k}(y_i) \dot{-} d_A(y_i) \\ &< (d_{R_k}(y_i) \dot{-} d(x_i, y_i)) + 2^{-k} \\ &\leq (d_{R_k}(y_i) \dot{-} d(r_i, y_i)) + 2^{-k} + 2^{-k} \\ &= 2^{-k+1}. \end{aligned}$$

It follows

$$d'_{\mathcal{A}}(A, R_k) \leq 2^{-k+1} + 2^{-j} |d_{R_k} \dot{-} d_A|_{K_{j-1}} + \sum_{i=j}^{k-2} 2^{-i-1} |d_{R_k} \dot{-} d_A|_{K_i}$$

$$
\begin{aligned}
&\le 2^{-k+1} + 2^{-j}|d_{R_k} \dot{-} d_A|_{K_j} + 2^{-j}2^{-k+1}\\
&\le 2^{-k+1} + 2^{-j}2^{-k+1} + 2^{-j}2^{-k+1}\\
&\le 2^{-k+1} + 2^{-k+2}\\
&\le 2^{-k+3}.
\end{aligned}
$$

(b) Let $\varepsilon > 0$ and $l \in \mathbb{N}$. Then there is some $j' \in \mathbb{N}$ such that $2^{-j'-1} < \varepsilon/2$ and $j' \ge j + l$. Then there is some $x \in A \cap K_{j'}$ and as in (a) one can show that there is some $r \in R_k$ for each $k > j'$ such that $d(x, r) < 2^{-k}$, i.e. $R_k \cap K_{j'+1} \ne \emptyset$ for all $k > j'$ since $d$ is nice. Since $d$ is nice we further obtain $|d_A \dot{-} d_{R_k}|_{K_i} = |d_A \dot{-} d_{R_k \cap K_{j'+1}}|_{K_i}$ for all $i = 0, ..., j'$ and $k > j'$ by Lemma 2.7.3. Now, for each $i = 0, ..., j'$ and $k > j'$ there is some $y_{ki} \in K_i$ such that $|d_A \dot{-} d_{R_k \cap K_{j'+1}}|_{K_i} = d_A(y_{ki}) \dot{-} d_{R_k \cap K_{j'+1}}(y_{ki})$ and there is some $r_{ki} \in R_k \cap K_{j'+1}$ such that $d_{R_k \cap K_{j'+1}}(y_{ki}) = d(r_{ki}, y_{ki})$. Since $r_{ki} \in R_k \subseteq D_{k-2k}$, we obtain $d'_{\mathcal{A}}(\{r_{ki}\}, S_{nk}) < 2^{-k+2}$ and thus

$$d'_{\mathcal{A}}(\{r_{ki}\}, A_n) \le d'_{\mathcal{A}}(\{r_{ki}\}, S_{nk}) + 2^{-k} < 2^{-k+3}$$

for all $n = 0, ..., m$, $i = 0, ..., j'$, $k > j'$. Consequently, due to Equation (2.3) we obtain $d_{A_n}(r_{ki}) \le 2^{j'+1} d'_{\mathcal{A}}(\{r_{ki}\}, A_n) < 2^{-k+j'+4}$ and there is some $x_{nki} \in A_n$ for each $n = 0, ..., m$ such that $d(x_{nki}, r_{ki}) < 2^{-k+j'+4}$. Since $K_{j'+1}$ is compact, there is a convergent subsequence $(r'_{ki})_{k>j'}$ of $(r_{ki})_{k>j'}$ for each $i = 0, ..., j'$. Let $(x'_{nki})_{k>j'}$ be the corresponding subsequence of $(x_{nki})_{k>j'}$. Then $\lim_{k\to\infty} x'_{nki} = \lim_{k\to\infty} r'_{ki} =: x_i$ and $x_i \in A = \bigcap_{n=0}^{m} A_n$ since $A_0, ..., A_m$ are closed, for each $i = 0, ..., j'$. Now for all $i = 0, ..., j'$ there is some $k' > j'$ such that $d_A(r'_{k'i}) \le d(x_i, r'_{k'i}) < \varepsilon/2$ and there is some $k > j' \ge j+l$ such that $r_{ki} = r'_{k'i}$. Thus,

$$
\begin{aligned}
d'_{\mathcal{A}}(R_k, A) &= \sum_{i=0}^{\infty} 2^{-i-1}|d_A \dot{-} d_{R_k}|_{K_i}\\
&\le 2^{-j'-1} + \sum_{i=0}^{j'} 2^{-i-1}|d_A \dot{-} d_{R_k}|_{K_i}\\
&\le \frac{\varepsilon}{2} + \sum_{i=0}^{j'} 2^{-i-1}(d_A(y_{ki}) \dot{-} d_{R_k \cap K_{j'+1}}(y_{ki}))\\
&= \frac{\varepsilon}{2} + \sum_{i=0}^{j'} 2^{-i-1}(d_A(y_{ki}) \dot{-} d_{r_{ki}}(y_{ki}))\\
&\le \frac{\varepsilon}{2} + \sum_{i=0}^{j'} 2^{-i-1}|d_A \dot{-} d_{r_{ki}}|_{K_i}
\end{aligned}
$$

$$\begin{aligned} &\le \frac{\varepsilon}{2} + \sum_{i=0}^{j'} 2^{-i-1} d_A(r'_{k'i}) \\ &< \varepsilon. \end{aligned}$$

□

Finally, we mention a corresponding result on the conjugate space.

**Proposition 3.9.4 (Conjugate space of closed subsets)** *If $(X, d, \alpha)$ is a nice recursively locally compact recursive metric space, then $(\mathcal{A}(X), \mathcal{A}(X), \overline{d'_{\mathcal{A}}}, \alpha_{\mathcal{A}})$ is a strong semi-recursive quasi-metric space.*

**Proof.** Using Theorem 2.7.7 and Lemma 3.9.2 this can be proved analogously to the corresponding Proposition 3.7.4 for compact subsets. □

Altogether we obtain the following result.

**Corollary 3.9.5 (Space of closed subsets)** *If $(X, d, \alpha)$ is a nice recursively locally compact recursive metric space, then $(\mathcal{A}(X), \mathcal{A}(X), d'_{\mathcal{A}}, \alpha_{\mathcal{A}})$ is a strong recursive quasi-metric space.*

By Theorem 3.5.1 we can even conclude that under the same assumptions $\delta_{\mathcal{A}} \equiv \delta_{\mathcal{A}_<} \sqcap \delta_{\mathcal{A}_>}$ holds for the corresponding representations. As we have mentioned in the previous section, $(\mathbb{R}^n, d_{\mathbb{R}^n}, \alpha_{\mathbb{R}^n})$ is a nice recursively locally compact recursive metric space with respect to the exhausting sequence $([-i, i]^n)_{i \in \mathbb{N}}$ and thus, by the previous theorem the corresponding space $(\mathcal{A}(\mathbb{R}^n), \mathcal{A}(\mathbb{R}^n), d'_{\mathcal{A}}, \alpha_{\mathcal{A}})$ is a semi-recursive quasi-metric space. Again we claim that the computable points of this space are well-known (cf. [BW99, BPar] and Appendix B). Here and in the following we write $\delta_{\mathcal{A}_<(X)}$ instead of $\delta_{\mathcal{A}(X)_<}$ and $\delta_{\mathcal{A}_>(X)}$ instead of $\delta_{\mathcal{A}(X)_>}$.

**Proposition 3.9.6** *The $\delta_{\mathcal{A}_<(\mathbb{R}^n)}$– and $\delta_{\mathcal{A}_>(\mathbb{R}^n)}$–computable points of the semi-recursive quasi-metric space $(\mathcal{A}(\mathbb{R}^n), \mathcal{A}(\mathbb{R}^n), d_{\mathcal{A}}, \alpha_{\mathcal{A}})$ are exactly the r.e. and co-r.e. subsets $A \subseteq \mathbb{R}^n$, respectively.*

We postpone the proof to Chapter 5 (it is a direct consequence of Theorem 5.1.8).

# Chapter 4

# Quasi-Metric Structures

## 4.1 Introduction

In the previous chapter we have seen how one can compute on quasi-metric spaces using the representation based approach to computable analysis. Now we would like to address the question how one can eliminate representations and describe computability by talking about points of quasi-metric spaces as entities. This question is directly related to the design of quasi-metric data structures. In the following we will consider many-sorted structures $(X_1, ..., X_n; f_1, ..., f_k)$ with a finite number of pairwise disjoint sets $X_1, ..., X_n$ and a finite number of initial operations $f_1, ..., f_k$. In his thesis [Bra99a] the author has presented a theory of perfect structures which have the pleasant property that the class of operations which can be constructed from the initial operations of such structures (using certain closure schemes) is exactly the class of computable operations (described by certain standard representations). In the following Sections 2 to 7 we will briefly recall some basic ideas of the theory of perfect structures. The presentation is based on [Bra02a] and all missing proofs can be found in [Bra99a, Bra02a]. In Section 8 and 9 we extend the theory of perfect structures to quasi-metric structures. The main result on quasi-metric structures is Theorem 4.9.1 which shows that any strong semi-recursive quasi-metric space can be endowed with a perfect structure which essentially contains the infimum operation and the quasi-metric as initial operations. Some rather technical parts of the proof of Theorem 4.9.1 are postponed to Appendix A. Using Theorem 4.9.1 and the results of the previous chapter, we derive a number of perfect quasi-metric structures for hyper and function spaces in Sections 10 to 12.

## 4.2 Computable multi-valued operations

In this section we want to present an extension of the representation based notion of computability to multi-valued operations. There are at least two motivations for such an extension. On the one hand, multi-valued operations occur in practice and examples like the determination of zeros of polynomials are important and unavoidable: given the coefficients of a non-constant polynomial, we can effectively find a zero, but in general the corresponding computation is non-extensional and therefore it has to be described by a multi-valued function.[1] Such *indeterministic computations*[2] may lead to different results on the same input, but all possible results have to be valid (in contrast to *non-deterministic* computations in complexity theory, where only some computations have to yield a valid result). On the other hand, one can motivate the introduction of multi-valued operations by formal reasons. If the arrows in the category of admissibly represented spaces are chosen to be multi-valued, then the category gains some nice properties, as we will implicitly see below.

Before we define computability of multi-valued operations, we introduce some notations. By $f :\subseteq X \rightrightarrows Y$ we will denote partial multi-valued functions which we will call for short *operations* in the following. Here the symbol "$\rightrightarrows$" indicates that $f$ might be multi-valued. More precisely, an operation $f :\subseteq X \rightrightarrows Y$ is a correspondence $f = (\Phi, X, Y)$, that is $\Phi \subseteq X \times Y$. We will use these objects from an operational point of view, that is $X$ is considered as a space of inputs and $Y$ as a space of outputs. We will use some notations for operations: $\text{graph}(f) := \Phi$, $\text{dom}(f) := \{x \in X : (\exists y \in Y)\ (x, y) \in \Phi\}$, and $\text{range}(f) := \{y \in Y : (\exists x \in X)\ (x, y) \in \Phi\}$ will be called *graph*, *domain*, and *range* of $f$, respectively. The *image* of $A \subseteq X$ under $f$ will be denoted by $f(A) := \{y \in Y : (\exists x \in A)\ (x, y) \in \Phi\}$, and the *preimage* of $B \subseteq Y$ by $f^{-1}(B) := \{x \in X : (\exists y \in B)\ (x, y) \in \Phi\}$. The *image* of $x$ under $f$ will be denoted by $f(x) := f\{x\} = \{y \in Y : (x, y) \in \Phi\}$ for each $x \in \text{dom}(f)$. If $f(x)$ is single-valued, i.e. $f(x) = \{y\}$ for some $y \in Y$, then we also write $f(x) = y$, as usual for functions. With each operation $f = (\Phi, X, Y)$ we associate the *inverse operation* $f^{-1} = (\Phi^{-1}, Y, X)$, with $\Phi^{-1} := \{(y, x) : (x, y) \in \Phi\}$.

Now we present our definition of computable operations which characterizes the complete image $f(x)$ by the help of an additional "oracle" input.

[1] The idea to compute with multi-valued operations has been considered by several authors in different settings [Mos69, EE70, Sko92, Kon00].

[2] This notion of indeterminism has already been used by Shepherdson [She75].

**Definition 4.2.1 (Computable operations)** Let $(X, \delta_X)$, $(Y, \delta_Y)$ be represented spaces. Then $f :\subseteq X \rightrightarrows Y$ is called a $(\delta_X, \delta_Y)$*–computable operation*, if there is a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that

$$\delta_Y F\langle p, \mathbb{N}^\mathbb{N}\rangle = f\delta_X(p)$$

and $\langle p, \mathbb{N}^\mathbb{N}\rangle \subseteq \mathrm{dom}(\delta_Y F)$ for all $p \in \mathrm{dom}(f\delta_X)$. Furthermore, $f$ is called a *strongly* $(\delta_X, \delta_Y)$*–computable operation*, if, additionally, the condition $\langle p, \mathbb{N}^\mathbb{N}\rangle \not\subseteq \mathrm{dom}(F)$ holds for all $p \in \mathrm{dom}(\delta_X) \setminus \mathrm{dom}(f\delta_X)$.

Here, $\langle p, \mathbb{N}^\mathbb{N}\rangle := \{\langle p, q\rangle : q \in \mathbb{N}^\mathbb{N}\}$. It is easy to see that the previous definition has the property that equivalent representations induce the same kind of (strong) computability of operations. Moreover, a single-valued function $f :\subseteq X \to Y$ is $(\delta_X, \delta_Y)$–computable, if and only if it is $(\delta_X, \delta_Y)$–computable, considered as an operation. And if $f$ is strongly $(\delta_X, \delta_Y)$–computable as a function, then it is also strongly $(\delta_X, \delta_Y)$–computable as an operation. However, the converse of the latter statement does not hold true in general. Therefore, in case of doubts, strong computability will always mean strong computability in the sense of operations.

We will close this section with a short discussion of continuity. It turns out that computable multi-valued operations are *lower semi-continuous*. As usual, a multi-valued operation $f :\subseteq X \rightrightarrows Y$ on topological spaces $X, Y$ is called *lower semi-continuous*,[3] if $f^{-1}(U)$ is open in $\mathrm{dom}(f)$ for any open subset $U \subseteq Y$. If $f$ is single-valued, then it is lower semi-continuous, if and only if it is continuous in the ordinary sense.

**Theorem 4.2.2 (Continuity)** *Let $(X, \delta_X)$, $(Y, \delta_Y)$ be admissibly represented second countable $T_0$–spaces. Then any $(\delta_X, \delta_Y)$–computable multi-valued operation $f :\subseteq X \rightrightarrows Y$ is lower semi-continuous.*

The proof can be found in [Bra02a].

## 4.3 Recursive closure schemes

In this section we discuss some set-theoretical closure schemes of computable multi-valued operations. These closure schemes are generalizations of the classical closure schemes *substitution*, *primitive recursion* and *$\mu$–recursion* [Odi89]. In the following we will assume that $U, V, X, Y, Z$ are arbitrary sets. All closure

[3] A corresponding notion of *upper semi-continuity* is defined with "closed" instead of "open" and in contrast to the single-valued case, upper semi-continuity is different from lower semi-continuity. Both concepts of semi-continuity should not be confused with the corresponding notions of continuity of single-valued real number functions.

schemes which will be introduced are defined for arbitrary sets, with exception of those places where the set $\mathbb{N}$ of natural numbers is mentioned explicitly.

**Definition 4.3.1 (Recursive closure schemes)** The following closure schemes are called *recursive closure schemes*:

(1) **Projection**: If $f :\subseteq X \rightrightarrows Y \times Z$ is an operation, then the *projection* $f_1 :\subseteq X \rightrightarrows Y$ is defined by

$$f_1(x) := \{y : (\exists z)\ (y, z) \in f(x)\}$$

for all $x \in \mathrm{dom}(f_1) := \mathrm{dom}(f)$. The projection $f_2 :\subseteq X \rightrightarrows Z$ on the second component is defined correspondingly.

(2) **Juxtaposition**: If $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq X \rightrightarrows Z$ are operations, then the *juxtaposition* $(f, g) :\subseteq X \rightrightarrows Y \times Z$ is defined by

$$(f, g)(x) := f(x) \times g(x) = \{(y, z) : y \in f(x) \text{ and } z \in g(x)\}$$

for all $x \in \mathrm{dom}(f, g) := \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

(3) **Product**: If $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq U \rightrightarrows V$ are operations, then the *product* $f \times g :\subseteq X \times U \rightrightarrows Y \times V$ is defined by

$$(f \times g)(x, u) := f(x) \times g(u) = \{(y, v) : y \in f(x) \text{ and } v \in g(u)\}$$

for all $(x, u) \in \mathrm{dom}(f \times g) := \mathrm{dom}(f) \times \mathrm{dom}(g)$.

(4) **Composition**: If $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$ are operations, then the *composition* $g \circ f :\subseteq X \rightrightarrows Z$ is defined by

$$(g \circ f)(x) := g(f(x)) := \{z : (\exists y \in f(x))\ z \in g(y)\}$$

for all $x \in \mathrm{dom}(g \circ f) := \{x : f(x) \subseteq \mathrm{dom}(g)\}$.

(5) **Iteration**: If $f :\subseteq X \rightrightarrows X$ is an operation, then the *iteration* $f^* :\subseteq X \times \mathbb{N} \rightrightarrows X$ is defined by

$$\begin{cases} f^*(x, 0) & := \{x\}, \\ f^*(x, n+1) & := f \circ f^*(x, n) \end{cases}$$

and abbreviated by $f^n(x) := f^*(x, n)$ for all $x \in X$ and $n \in \mathbb{N}$.

(6) **Inversion**: If $f :\subseteq X \times \mathbb{N} \rightrightarrows Y \times \mathbb{N}$ is an operation, then the (*twisted*) *inversion* $f^{\leftrightarrow} :\subseteq X \times \mathbb{N} \rightrightarrows Y \times \mathbb{N}$ is defined by

$$f^{\leftrightarrow}(x, n) := \{(y, k) : (y, n) \in f(x, k)\}$$

for all $(x, n) \in \mathrm{dom}(f^{\leftrightarrow}) := \{(x, n) : (\forall k)\ (x, k) \in \mathrm{dom}(f)$ and $(\exists k)\ n \in f_2(x, k)\}$.

(7) **Evaluation**: If $f :\subseteq X \rightrightarrows Y^{\mathbb{N}}$ is an operation, then the *evaluation* $f_* :\subseteq X \times \mathbb{N} \rightrightarrows Y$ is defined by

$$f_*(x,n) := \{y : (\exists (y_k)_{k\in\mathbb{N}} \in f(x))\ y_n = y\}$$

for all $(x,n) \in \mathrm{dom}(f_*) := \mathrm{dom}(f) \times \mathbb{N}$.

(8) **Transposition**: If $f :\subseteq X \times \mathbb{N} \rightrightarrows Y$ is an operation, then the *transposition* $[f] :\subseteq X \rightrightarrows Y^{\mathbb{N}}$ is defined by

$$[f](x) := \{(y_n)_{n\in\mathbb{N}} : (\forall n)\ y_n \in f(x,n)\}$$

for all $x \in \mathrm{dom}([f]) := \{x : (\forall n)\ (x,n) \in \mathrm{dom}(f)\}$.

(9) **Exponentiation**: If $f :\subseteq X \rightrightarrows Y$ is an operation, then the *exponentiation* $f^{\mathbb{N}} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ is defined by

$$f^{\mathbb{N}}((x_n)_{n\in\mathbb{N}}) := \{(y_n)_{n\in\mathbb{N}} : (\forall n)\ y_n \in f(x_n)\}$$

for all $(x_n)_{n\in\mathbb{N}} \in \mathrm{dom}(f^{\mathbb{N}}) := \{(x_n)_{n\in\mathbb{N}} : (\forall n)\ x_n \in \mathrm{dom}(f)\}$.

(10) **Sequentialization**: If $f :\subseteq X \rightrightarrows \mathbb{N}$ is an operation, then the *sequentialization* $f^{\triangle} :\subseteq X \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is defined by

$$f^{\triangle}(x) := \{(y_n)_{n\in\mathbb{N}} : f(x) = \{y_n : n \in \mathbb{N}\}\}$$

for all $x \in \mathrm{dom}(f^{\triangle}) := \mathrm{dom}(f)$.

The closure schemes projection, juxtaposition and product can be used to handle product spaces and analogously, evaluation, transposition and exponentiation can be used to handle sequence spaces (the latter three schemes are infinite versions of the former three). In presence of projection, juxtaposition and product, the classical schemes of substitution, primitive recursion and minimization can be realized by composition, iteration and inversion. Finally, sequentialization is a scheme which can be used to eliminate indeterminism in certain cases.[4]

**Theorem 4.3.2** *On topological spaces, all recursive closure schemes preserve lower semi-continuity of multi-valued operations.*

[4]The definitions of most of the closure schemes are straightforward, although the details need some care. For instance, a second natural choice for the domain $\mathrm{dom}(g \circ f)$ of the composition of $g$ and $f$ would be the set $\{x : f(x) \cap \mathrm{dom}(g) \neq \emptyset\}$. However, this definition would not satisfy Theorem 4.3.2.

A proof can be found in [Bra99a]. Using the notations of Definition 4.3.1, we tacitly assume for the previous result that $U, V, X, Y, Z$ are topological spaces and $\mathbb{N}$ is endowed with the discrete topology. Moreover, product and sequence spaces are to be endowed with the corresponding product topologies. Actually, the closure schemes do not only preserve continuity but computability as well. For the following result we assume that $(U, \delta_U)$, $(V, \delta_V)$, $(X, \delta_X)$, $(Y, \delta_Y)$, $(Z, \delta_Z)$ are represented spaces and that $\mathbb{N}$ is represented by $\delta_{\mathbb{N}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ with $\delta_{\mathbb{N}}(p) := p(0)$ for all $p \in \mathbb{N}^{\mathbb{N}}$. Product spaces and sequence spaces are represented by the corresponding product and sequence representations.

**Theorem 4.3.3** *On represented spaces, all recursive closure schemes preserve (strong) computability of multi-valued operations.*

## 4.4 Recursive operations over structures

In this section we will use the recursive closure schemes to introduce the notion of a recursive operation over a structure. We will consider many-sorted structures $(X_1, ..., X_n; f_1, ..., f_k)$ with a finite number of pairwise disjoint sets $X_1, ..., X_n$ and a finite number of operations $f_1, ..., f_k$. We start with the definition of a *prestructure* (prestructures have the property that the operations are not necessarily related to the universes).

**Definition 4.4.1 (Prestructures)** $S = (X_1, ..., X_n; f_1, ..., f_k)$ is called a *many-sorted prestructure* with *universe* $X_1 \times ... \times X_n$, if $X_1, ..., X_n$ are pairwise disjoint sets and $f_1, ..., f_k$ are arbitrary operations, called the *initial operations* of $S$.

Usually, we will say for short *prestructure* instead of many-sorted prestructure. If $S = (X_1, ..., X_n; f_1, ..., f_k)$ is a prestructure, then we consider the sets $X_1, ..., X_n$ as *atomic* (in the sense that, as long as structures are concerned, we ignore the fact that these sets might be defined themselves as products of other sets). In the next step we define the notion of a set over a prestructure $S = (X_1, ..., X_n; f_1, ..., f_k)$. Roughly speaking, a set over a prestructure $S$ is a set that can be constructed as product or sequence set of the sets $X_1, ..., X_n$. This definition takes into account that we want to handle product spaces as well as spaces of sequences.

**Definition 4.4.2 (Sets and operations over prestructures)** The *class of sets over* a prestructure $S = (X_1, ..., X_n; f_1, ..., f_k)$ is the smallest class of sets such that:

(1) $X_1, ..., X_n$ and $\{()\}$ are sets over $S$,

(2) $X \times Y$ and $X^{\mathbb{N}}$ are sets over $S$, if $X, Y$ are sets over $S$.

Moreover, $f :\subseteq X \rightrightarrows Y$ is called an *operation over* $S$, if $X, Y$ are sets over $S$.

Here, the special set $\{()\}$ occurs which has the *empty tuple* $()$ as single element. We assume $X^0 = \{()\}$ for any set $X$ and we will always endow $\{()\}$ with the discrete topology. We use the representation $\delta : \mathbb{N}^{\mathbb{N}} \to \{()\}$, defined by $\delta(p) := ()$ for all $p \in \mathbb{N}^{\mathbb{N}}$. If we identify $X$ with $\{()\} \times X$ and $X \times \{()\}$, then we obtain $\delta' \equiv [\delta, \delta'] \equiv [\delta', \delta]$ for any representation $\delta'$ of $X$. Thus, this identification will lead to no complications with respect to computability and continuity and we will tacitly apply it (in particular, when using the recursive closure schemes[5]). Now we will single out those prestructures $S$ whose initial operations are operations over $S$ and we will call them *structures*.

**Definition 4.4.3 (Structures)** A prestructure $S$ is called a *structure*, if all initial operations of $S$ are operations over $S$.

The reason why the universe of a structure $S$ is defined as product is a technical one: it ensures that the universe itself is a set over $S$. If $S = (X_1, ..., X_n; f_1, ..., f_k)$ and $T = (Y_1, ..., Y_m; g_1, ..., g_l)$ are structures such that $Y_1, ..., Y_m \in \{X_1, ..., X_n\}$ and $g_1, ..., g_l \in \{f_1, ..., f_k\}$, then $T$ is called a *substructure* of $S$ and we write $T \sqsubseteq S$. Now we proceed to define recursive operations over structures.

**Definition 4.4.4 (Recursive operations over structures)** The class of *recursive operations over a structure* $S$ is the smallest class of operations which contains all initial operations of $S$ and which is closed under the recursive closure schemes.

From the definition of the closure schemes it is obvious that each recursive operation over a structure is an operation over that structure.

A constant $x \in X$ is called a *recursive constant over a structure* $S$, if the operation $\{()\} \to X, () \mapsto x$ is recursive over $S$. If there is a constant $c \in X$ among the initial operations of a structure, then we will consider it as zero-ary constant function $X^0 \to X, () \mapsto c$ with value $c$.

[5]More formally, we could add closure schemes which allow to add and delete the set $\{()\}$ as factor in the target or source.

A very important structure is the *structure of natural numbers* which is given by

$$\mathbf{N} := (\mathbb{N}; 0, n, n+1).$$

By "0" we denote the *zero-ary constant function* $\mathbb{N}^0 \to \mathbb{N}, () \mapsto 0$ with value 0, by "$n$" the *identity* $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}, n \mapsto n$, and by "$n+1$" the *successor function* $S : \mathbb{N} \to \mathbb{N}, n \mapsto n+1$. Here and in the following we will often use bold letters, as "$\mathbf{N}$", to distinguish a structure from its universe. It is routine to prove the following result which shows that the class of recursive operations over $\mathbf{N}$ is quite rich.

**Proposition 4.4.5** *Each computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is recursive over $\mathbf{N}$.*

One can even prove this result without using the closure scheme of sequentialization, see [Bra99a]. In the following we will often need structures which include the natural numbers and some essential operations. These structures will be called *natural*. By $\mathrm{id}_X : X \to X, x \mapsto x$ we denote the *identity* of $X$.

**Definition 4.4.6 (Natural structures)** A structure $S$ with universe $X$ is called *natural*, if $\mathbf{N}$ is a substructure of $S$, $\mathrm{id}_X$ is recursive over $S$ and there is a recursive constant $c \in X$ over $S$.

It is easy to see that for natural structures $S = (X_1, ..., X_n; f_1, ..., f_k)$ the identity of each set over $S$ is recursive. Especially, the identities $\mathrm{id}_{X_1}, ..., \mathrm{id}_{X_n}$ are recursive, and each set $X_1, ..., X_n$ contains at least one recursive constant.

Sometimes it is also suitable to consider structures where a special constant operation is available. For each set $A \subseteq X$ we define the *omnipotent operation* $\Omega_A : \{()\} \rightrightarrows X, () \mapsto A$ of $A$, i.e. $\Omega_A$ is the constant operation which yields as result $\Omega_A() = A$.

**Definition 4.4.7 (Complete structure)** A structure $S$ with universe $X$ is called a *complete structure*, if $\Omega_X$ is recursive over $S$.

Since $\Omega_{\mathbb{N}} = z^{\leftrightarrow} \circ 0$, where $z = (0 \times \mathrm{id}_{\mathbb{N}})_1$ denotes the constant zero function $z : \mathbb{N} \to \mathbb{N}, n \mapsto 0$ and $\Omega_{\mathbb{N}^{\mathbb{N}}} = [(\Omega_{\mathbb{N}} \times \mathrm{id}_{\mathbb{N}})_1]$, we obtain that $\Omega_{\mathbb{N}}$ and $\Omega_{\mathbb{N}^{\mathbb{N}}}$ are recursive over each natural structure. Especially, the structure $\mathbf{N}$ is complete. The reader should notice that we identify $\mathbb{N}^0 \times \mathbb{N}$ with $\mathbb{N}$, as mentioned above.

Our next goal is to state that recursive operations are also computable. Therefore, we will first define representations of structures and computability on structures.

**Definition 4.4.8 (Representations of structures)** Consider a structure $S = (X_1, ..., X_n; f_1, ..., f_k)$ and let $\delta_1, ..., \delta_n$ be representations of $X_1, ..., X_n$, respectively. Then $\delta := [\delta_1, ..., \delta_n]$ is called a *representation* of $S$ and $(S, \delta)$ is called a *represented structure*.

Using this definition we can extend the notion of computability to structures.[6]

**Definition 4.4.9 (Computability on structures)** Consider a structure $S = (X_1, ..., X_n; f_1, ..., f_k)$ with representation $\delta = [\delta_1, ..., \delta_n]$. Let $f :\subseteq Y \rightrightarrows Z$ be an operation over $S$ and let $\delta_Y, \delta_Z$ be representations of $Y, Z$, respectively, which are finitely generated from $\delta_1, ..., \delta_n$, correspondingly as $Y, Z$ are finitely generated from $X_1, ..., X_n$. Then $f$ is called (*strongly*) *computable* with respect to $\delta$, if it is (strongly) $(\delta_Y, \delta_Z)$–computable.

It should be clear how the representations $\delta_Y, \delta_Z$ have to be constructed correspondingly to $Y, Z$ (we recall that we consider the sets $X_1, ..., X_n$ as atomic). If, for instance, $Y = (X_1 \times X_2)^{\mathbb{N}}$, then the corresponding representation is $\delta_Y = [\delta_1, \delta_2]^\infty$ (see Section 3.1 for the definitions of $[\delta_1, \delta_2]$ and $\delta^\infty$). In the next step we define the notion of an effective structure.

**Definition 4.4.10 (Effective structures)** A structure $S$ is called a (*strongly*) *effective structure*, if there is a representation $\delta$ of $S$ such that all initial operations of $S$ are (strongly) computable with respect to $\delta$. In this situation $S$ is also called (*strongly*) *effective via* $\delta$.

It is easy to see that the structure **N** is strongly effective via $\delta_{\mathbb{N}}$. In the following we will use effective structures where the effectivity on natural numbers is fixed.

**Definition 4.4.11 (Natural representations)** If $S = (X_1, ..., X_n; f_1, ..., f_k)$ is a natural structure, then we will say that a representation $\delta = [\delta_1, ..., \delta_n]$ of $S$ is a *natural representation*, if $X_i = \mathbb{N}$ implies $\delta_i \equiv \delta_{\mathbb{N}}$ for all $i = 1, ..., n$.

By structural induction we obtain the following corollary of Theorem 4.3.3.

**Corollary 4.4.12 (Recursive operations over effective structures)** *If $S$ is a natural structure which is (strongly) effective via a natural representation $\delta$, then each operation which is recursive over $S$ is also (strongly) computable with respect to $\delta$.*

[6] Here and in the following all bracket terms "(strongly)" are to be read together or, alternatively, they all have to be omitted.

## 4.5 Perfect structures

In this section we introduce the class of recursive structures and we will prove that over recursive structures each strongly computable operation is also recursive. This leads us to the definition of perfect structures which are both: effective and recursive. Perfect structures have several nice properties, especially they are categorical in the sense that they characterize their own computability theory. In order to define recursive structures, we will use the notion of a recursive retraction.

**Definition 4.5.1 (Recursive retraction)** An operation $f :\subseteq X \rightrightarrows Y$ over a structure $S$ is called a *recursive retraction* over $S$, if it is recursive and if it admits a recursive right inverse operation $f^- : Y \rightrightarrows X$ over $S$, i.e. $f \circ f^- = \mathrm{id}_Y$.

If $f$ is a recursive retraction, then the restriction of $f$ to $\mathrm{range}(f^-)$ is a surjective function. A structure will be called strongly recursive, if it admits a representation which is a recursive retraction.

**Definition 4.5.2 (Recursive structure)** Let $S$ be a structure. Then

(1) $S$ is called a *recursive structure*, if there is a representation $\delta$ of $S$, which admits a recursive extension, as well as a recursive right inverse operation over $S$,

(2) $S$ is called a *strongly recursive structure*, if there is a representation $\delta$ of $S$ which is a recursive retraction over $S$.

In these cases $S$ is also called *(strongly) recursive via* $\delta$.

Intuitively, a strongly recursive structure is a structure with a representation which can be "synthesized" as well as "analyzed" within the structure. Now we mention that the structure of natural numbers is recursive.

**Proposition 4.5.3** *The structure* $\mathbf{N}$ *is strongly recursive via* $\delta_{\mathbb{N}}$.

Since the structure of natural numbers is recursive it makes sense to apply the notion of recursiveness to natural structures. Now we can state the announced converse version of Corollary 4.4.12.

**Theorem 4.5.4 (Computable operations over recursive structures)** *If* $S$ *is a natural structure which is recursive via a representation* $\delta$*, then each operation over* $S$ *which is computable with respect to* $\delta$ *also admits a recursive extension over* $S$*.*

Analogously, if $S$ is a natural structure which is *strongly* recursive via a representation $\delta$, then each operation over $S$ which is *strongly* computable with respect to $\delta$ is also recursive over $S$. The proof is based on the following Lemma which we mention since it will be applied later for different purposes.

**Lemma 4.5.5** *Let $S$ be a natural structure and let $\delta_i :\subseteq \mathbb{N}^\mathbb{N} \to X_i$, $i = 1, ..., n$, $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$ be representations. If $\delta_1, ..., \delta_n, \delta$ are recursive retractions over $S$, then so are $[\delta_1, ..., \delta_n]$ and $\delta^\infty$.*

We have seen that over effective structures recursive operations are computable and over recursive structures computable operations are recursive. Thus, it makes sense to consider structures which are both, recursive and effective. We will see that such structures have very nice properties. The first result states that for such structures there is a unique effectivity which is characterized by the structure itself.

**Theorem 4.5.6 (Stability theorem)** *If $S$ is a natural structure which is recursive via a representation $\delta$ and which is effective via a natural representation $\delta'$, then $\delta \equiv \delta'$.*

We can deduce that, if a structure $S$ is recursive, then all natural representations $\delta$ such that $S$ is effective via $\delta$ are equivalent. Thus, if $S$ is recursive and effective, then it is *effectively categorical*, a notion that has been used in a similar setting by Hertling [Her99]. But even more, if $S$ is effective via a natural representation, then all representations $\delta$ which are recursive retractions over $S$ are also equivalent. Thus, if $S$ is recursive and effective, then it has a further property which could be called *recursively categorical*.

It should be mentioned that there is no hope for a corresponding result without the restriction to natural representations. As long as we do not demand any evaluation property for the output, we can effectivize a structure just by using terms and their evaluations. The situation changes if we do fix effectivity for at least one "output sort". For instance, in the classical setting this can be done by fixing effectivity for the boolean sort, i.e. the "output sort" of predicates.[7] In our setting, we have fixed effectivity on the natural numbers. The Stability Theorem gives rise to the following definition.

**Definition 4.5.7 (Perfect structures)** A natural structure $S$ is called (*strongly*) *perfect*, if it is (strongly) recursive and (strongly) effective via a natural representation. In this situation each natural representation $\delta$ such

[7]This is the case in Mal'cev's Stability Theorem for finitely generated algebras [Mal71, SHT95], which can be considered as a special version of the Stability Theorem 4.5.6, see [Bra99b].

that $S$ is (strongly) recursive via $\delta$ is called a (*strong*) *standard representation* of $S$.

It is easy to see that **N** is a strongly perfect structure with strong standard representation $\delta_{\mathbb{N}}$. The Stability Theorem states that perfect structures uniquely characterize their effectivity. Especially, all standard representations of a perfect structure and all natural representations which make this structure effective belong to the same equivalence class. Therefore, we can define computability over perfect structures without mentioning any special representation.

**Definition 4.5.8 (Comp. operations over perfect structures)** An operation $f$ over a perfect structure $S$ is called (*strongly*) *computable over* $S$, if it is (strongly) computable with respect to a standard representation $\delta$ of $S$.

Now we can formulate a corollary of Theorem 4.5.4 and Corollary 4.4.12 which characterizes recursive operations over (strongly) perfect structures.

**Theorem 4.5.9 (Operations over perfect structures)** *An operation over a strongly perfect structure is strongly computable, if and only if it is recursive. An operation over a perfect structure is computable, if and only if it admits a recursive extension.*

Finally, we obtain an Extension Theorem for operations over strongly perfect structures as a combination of both results.

**Corollary 4.5.10 (Extension Theorem)** *Each computable operation over a strongly perfect structure admits a strongly computable extension.*

Another nice property of strongly perfect structures is the property of *conservative extension*. Consider a natural structure $S$ with an *extension* $S'$, i.e. a structure $S'$ such that $S \sqsubseteq S'$. In general, the additional initial operations of $S'$ could increase the class of recursive operations even over the universe of $S$. The following theorem states that this cannot happen over perfect structures.

**Theorem 4.5.11 (Conservation Theorem)** *Let $S \sqsubseteq S'$ be perfect structures and let $f$ be an operation over $S$. Then $f$ admits a recursive extension over $S$, if and only if it admits a recursive extension over $S'$.*

Analogously, if $S \sqsubseteq S'$ are strongly perfect structures and if $f$ is an operation over $S$, then $f$ is recursive over $S$, if and only if $f$ is recursive over $S'$.

In the following, by $S \oplus T := (X_1, ..., X_n, Y_1, ..., Y_m; f_1, ..., f_k, g_1, ..., g_l)$ we will denote the *union* of a prestructure $S = (X_1, ..., X_n; f_1, ..., f_k)$ with a prestructure $T = (Y_1, ..., Y_m; g_1, ..., g_l)$. If there are double occurrences among the sets or operations, then just the first occurrence (from the left) is kept, e.g. $(\mathbf{N} \oplus T) \oplus (\mathbf{N} \oplus S) = \mathbf{N} \oplus (T \oplus S)$. If there are no such double occurrences, then we say that $S$ and $T$ are *disjoint prestructures*.

**Theorem 4.5.12 (Union)** *Let $R, S$ and $T$ be pairwise disjoint prestructures. If $R \oplus S$ and $R \oplus T$ are (strongly) perfect natural structures, then their union $R \oplus (S \oplus T)$ is a (strongly) perfect natural structure.*

We close this section with a short meta-analysis of our results. We have already mentioned that for the proof of Proposition 4.4.5 it suffices to use all recursive closure schemes besides sequentialization. Proposition 4.4.5 has been used for the proof of Theorem 4.5.4. But neither for this theorem nor for the other results of this section we have used the sequentialization operator. Indeed, all results on perfect structures remain true, if we exclude sequentialization from our recursive closure schemes. However, the class of perfect structures could be smaller in this case and this is the reason why we have included sequentialization.

## 4.6 Topological and metric structures

In the previous section we have seen that perfect structures have some nice properties but we have not proved that there are any interesting perfect structures. In this section we will show that $T_0$–spaces with countable bases admit perfect topological structures and we will define a perfect standard structure for separable metric spaces. We start with the definition of the notion of a topological structure. It is natural to assume that the universes of such structures are topological spaces and that the initial operations are continuous.

**Definition 4.6.1 (Topological structure)** A *topological structure* is a structure $S = (X_1, ..., X_n; f_1, ..., f_k)$ where $X_1, ..., X_n$ are endowed with topologies and all initial operations $f_1, ..., f_k$ are lower semi-continuous with respect to the corresponding product topologies. Moreover, $S$ is called *natural topological structure*, if $S$ is a natural structure and the natural numbers come equipped with the discrete topology.

If $Y$ is a set over a structure $S$, then it inherits a product topology which is generated from the topologies of $X_1, ..., X_n$, correspondingly as $Y$ is generated from the sets $X_1, ..., X_n$. In this situation we will say that $Y$ is a *topological space over $S$*. In the following we will sometimes use short names for topological structures with additional properties. For instance, a “metric structure” will be a topological structure where the universe is endowed with a metric. As we have seen in Theorem 4.3.2, all recursive closure schemes preserve lower semi-continuity. Therefore, we obtain immediately the following corollary.

**Corollary 4.6.2 (Continuity of recursive operations)** *All recursive operations over natural topological structures are lower semi-continuous.*

If $X$ is a second countable $T_0$–space, then there exists an open and continuous representation $\delta$ of $X$ (see Definition 5.3.8), hence $\delta$, as well as its inverse $\delta^{-1} : X \rightrightarrows \mathbb{N}^{\mathbb{N}}$, are lower semi-continuous. If there is some computable point $p \in \mathrm{dom}(\delta)$, then $\mathbf{X} = \mathbf{N} \oplus (X; \delta, \delta^{-1})$ is a natural topological structure. Obviously, $\delta' := [\delta_{\mathbb{N}}, \delta]$ is a representation of $\mathbf{X}$ and $\mathbf{X}$ is recursive via $\delta'$. Moreover, $\mathbf{X}$ is also effective via $\delta'$ since $\delta$ is $(\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, \delta)$–computable and $\delta^{-1}$ is $(\delta, \mathrm{id}_{\mathbb{N}^{\mathbb{N}}})$–computable. Altogether we obtain the following result.

**Theorem 4.6.3 (Perfect topological structures)** *If $X$ is a second countable $T_0$–space with open and continuous representation $\delta$ such that there is some computable point in $\mathrm{dom}(\delta)$, then $\mathbf{X} = \mathbf{N} \oplus (X; \delta, \delta^{-1})$ is a perfect topological structure with standard representation $[\delta_{\mathbb{N}}, \delta]$.*

Here, second countability is not only a sufficient but even a necessary condition. If $\mathbf{X}$ is a natural topological structure which is recursive via some representation $\delta$, then there exists some right inverse $\delta^-$ of $\delta$ which is recursive over $\mathbf{X}$. Then $\{(\delta^-)^{-1}(w\mathbb{N}^{\mathbb{N}}) : w \in \mathbb{N}^*\}$ is a countable base of $X$. Thus, we have proved the following result.

**Proposition 4.6.4** *If $S$ is a recursive topological structure with universe $X$, then the topology of $X$ has a countable base.*

By Theorem 4.6.3 we know that any second countable $T_0$–space admits a perfect topological structure, but the structure given in Theorem 4.6.3 is quite artificial because it contains a representation and its inverse as initial operations. In the following we will see that separable metric spaces admit far more natural structures. As a special case we first investigate a structure for the real numbers. As usual, 0 and 1 denote the corresponding real constants, $+, -, \cdot, /$ the usual arithmetic operations on the real numbers with their natural domains. By $<$ we denote the *semi-characteristic operation* of the usual order relation:

$$c_< : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{N}, (x, y) \mapsto \begin{cases} \{0, 1\} & \text{if} \quad x < y \\ \{1\} & \text{else} \end{cases}$$

This definition will be generalized and discussed below. As defined before, $\mathrm{Lim} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ denotes the *limit operator* (restricted to rapidly converging sequences) with respect to the Euclidean distance.

**Theorem 4.6.5 (The structure of the real numbers)** *The structure*

$$\mathbf{R} := \mathbf{N} \oplus (\mathbb{R}; 0, 1, x+y, -x, x \cdot y, 1/x, \mathrm{Lim}, x < y)$$

*is a strongly perfect complete topological structure and the Cauchy representation yields a standard representation of this structure.*

More precisely, $[\delta_{\mathbb{N}}, \delta_{\mathbb{R}}]$ is a standard representation of $\mathbf{R}$. As a direct consequence of this theorem and Theorem 4.5.9, we can deduce that the recursive functions over $\mathbf{R}$ are exactly the strongly computable ones. As a special case we obtain that the total recursive functions $f : \mathbb{R}^n \to \mathbb{R}$ over $\mathbf{R}$ are exactly the classically computable functions (according to Grzegorczyk's and Lacombe's definitions [Grz55, Lac55]).

**Corollary 4.6.6** *The recursive functions $f : \mathbb{R}^n \to \mathbb{R}$ over $\mathbf{R}$ are exactly the classically computable real functions.*

As we will see in the following, the perfectness result on $\mathbf{R}$ can be generalized to recursive metric spaces. As usual, we denote by $\mathrm{Lim} :\subseteq X^{\mathbb{N}} \to X$ the limit operation (restricted to rapidly converging sequences).

**Theorem 4.6.7 (Metric structures)** *If $(X, d, \alpha)$ is a (complete) recursive metric space, then*

$$\mathbf{X} := \mathbf{R} \oplus (X; \alpha, \mathrm{id}, d, \mathrm{Lim})$$

*is a (strongly) perfect (complete) topological structure.*[8] *The Cauchy representation yields a standard representation of this structure.*

More precisely, if $\delta_X$ is the Cauchy representation of the space $X$, then $[\delta_{\mathbb{N}}, \delta_{\mathbb{R}}, \delta_X]$ is a standard representation of $\mathbf{X}$. The proof is based on Proposition 3.1.4 and a generalization of the proof of Theorem 4.6.5. The structure $\mathbf{X}$ is already a bit more natural than the structure given in Theorem 4.6.3. However, in many concrete cases it is even possible to "decompose" $\alpha$ with the help of the algebraic structure of the given space. We have seen this already in case of the real number structure $\mathbf{R}$. We give some further examples. Again, we will use a $\lambda$-like notation for operations, the details can be found in the Table of Perfect Structures (in the Appendix).

[8] Here and in the following all bracket terms "(strongly)" or "(complete)" are to be read together or, alternatively, they all have to be omitted.

**Example 4.6.8 (Perfect metric structures)**

(1) If $(X,d)$ is a (complete) recursive metric space, then the structure

$$\mathcal{K}(\mathbf{X}) := \mathbf{X} \oplus (\mathcal{K}(X); \{x\}, A, A\cup B, d_{\mathcal{K}}, \mathrm{Lim})$$

is a (strongly) perfect (complete) topological structure with respect to the Vietoris topology, generated by the Hausdorff metric

$$d_{\mathcal{K}} : \mathcal{K}(X)\times\mathcal{K}(X) \to \mathbb{R}, (A,B) \mapsto \max\left\{\sup_{a\in A} d_B(a), \sup_{b\in B} d_A(b)\right\}.$$

(2) If $X$ is a recursively locally compact recursive metric space with recursive exhausting sequence$(K_i)_{i\in\mathbb{N}}$, then the structure

$$\mathcal{C}(\mathbf{X}) := \mathbf{X} \oplus (\mathcal{C}(X); 1, d_x, f, y\cdot f, f+g, f\cdot g, d_{\mathcal{C}}, \mathrm{Lim})$$

is a strongly perfect complete topological structure with respect to the compact open topology, induced by the metric

$$d_{\mathcal{C}} : \mathcal{C}(X)\times\mathcal{C}(X) \to \mathbb{R}, (f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1}\frac{|f-g|_{K_i}}{1+|f-g|_{K_i}}.$$

(3) If $X$ is a nice recursively locally compact recursive metric space with recursive exhausting sequence$(K_i)_{i\in\mathbb{N}}$, then the structure

$$\mathcal{A}(\mathbf{X}) := \mathbf{X} \oplus (\mathcal{A}(X); \{x\}, A, A\cup B, d_{\mathcal{A}}, \mathrm{Lim})$$

is a strongly perfect topological structure with respect to the Fell topology, induced by the metric

$$d_{\mathcal{A}} : \mathcal{A}(X)\times\mathcal{A}(X) \to \mathbb{R}, (A,B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1}|d_B - d_A|_{K_i}.$$

Proofs and a detailed discussion of further examples can be found in [Bra99a]. By the Stability Theorem 4.5.6 the results of this section especially show that the structures **R** and **X** are effectively categorical. This has already been proved in a slightly different setting by Hertling [Her99]. By results of Hemmerling [Hem01] it follows that effectively categorical structures for recursive metric spaces necessarily include infinitary operations (as Lim). Since second countable $T_0$–spaces are quasi-metrizable, it is promising to transfer the considerations of this section to semi-recursive quasi-metric spaces, as we will do below.

## 4.7 Recursive sets over structures

In this section we briefly discuss some effectivity notions for sets over structures. If $A \subseteq X$, then we sometimes write $A^c := X \setminus A$ for the *complement* of the set $A$.

**Definition 4.7.1 (Recursive sets over structures)** Let $S$ be a natural structure, let $X$ be a set over $S$ and let $A \subseteq X$.

(1) $A$ is *initially semi-recursive* over $S$, if there exists a total recursive operation $f : X \rightrightarrows \mathbb{N}$ over $S$ such that $A = f^{-1}\{0\}$.

(2) $A$ is *finally semi-recursive* over $S$, if there exists a partial recursive operation $f :\subseteq \mathbb{N} \rightrightarrows X$ over $S$ such that $A = f\{0\}$.

(3) $A$ is *recursive* over $S$, if $A$ is finally semi-recursive and $X \setminus A$ is initially semi-recursive over $S$.

(4) $A$ is *decidable* over $S$, if there exists a total recursive function $f : X \to \mathbb{N}$ over $S$ such that $A = f^{-1}\{0\}$.

In the following we will say for short *semi-recursive* instead of initially semi-recursive. Obviously, the empty set $\emptyset$, considered as a subset of a set $X$ over a natural structure $S$, is always recursive and decidable. Furthermore, the set $X$, considered as a subset of itself, is always initially semi-recursive and decidable over $S$ and $X$ is also finally semi-recursive and hence recursive over $S$, if and only if $\Omega_X$ is recursive over $S$. In general, it is easy to see that over a natural structure $S$ a non-empty subset $A$ is finally semi-recursive, if and only if $\Omega_A$ is recursive over $S$. Especially, all sets over complete structures are finally semi-recursive and a point $x \in X$ is recursive over $S$, if and only if $\{x\}$ is finally semi-recursive over $X$. Over natural structures $S$ a subset $A$ is decidable, if and only if its *characteristic function* $\mathrm{cf}_A$ is recursive over $S$. We introduce the *semi-characteristic operation* to characterize semi-recursive sets $A$ in a corresponding way.

**Definition 4.7.2 (Semi-characteristic operation)** For each set $X$ and each subset $A \subseteq X$ we define the *semi-characteristic operation* $c_A : X \rightrightarrows \mathbb{N}$ of $A$ by

$$c_A(x) := \begin{cases} \{0,1\} & \text{if } x \in A \\ \{1\} & \text{else} \end{cases}$$

With this notation a subset $A$ over a natural structure $S$ is semi-recursive, if and only if $c_A$ is recursive over $S$. The intuition behind a semi-characteristic operation $c_A$ is that it operates as follows: for any input $x \in X$ the operation can answer “no” (indicated by the value 1), but exactly for all inputs $x \in A$ there has to be a computation such that the operation yields the answer “yes” (indicated by the value 0).

It is easy to see that semi-recursive and decidable sets are closed under intersection and semi-recursive, finally semi-recursive, recursive and decidable sets are closed under union. Decidable subsets are obviously closed under complement as well and, as in the classical case we obtain that a subset $A \subseteq X$ of a set $X$ over a natural structure $S$ is decidable over $S$, if and only if $A$ and $A^c$ are semi-recursive over $S$. Moreover, semi-recursive sets are closed under preimage of recursive operations and finally semi-recursive sets under the image (provided they are contained in the domain). Over complete natural structures, semi-recursiveness implies final semi-recursiveness.

**Proposition 4.7.3** *Over complete natural structures each semi-recursive set is also finally semi-recursive and each decidable subset is recursive.*

For any operation $f :\subseteq X \rightrightarrows Y \times \mathbb{N}$ we call $f_0 :\subseteq X \rightrightarrows Y$, defined by $f_0 := ((f \circ \mathrm{pr}_1)^{\leftrightarrow})_1 \circ (\mathrm{id}_X \times 0)$, the *section* of $f$. This name is justified since one obtains $f_0(x) = \{y \in Y : (y, 0) \in f(x)\}$. The proof of the proposition follows immediately from $\Omega_A = ((\mathrm{id}_X, c_A) \circ \Omega_X)_0$. The next important property is an “uniformization property” which states that semi-recursive sets are graphs of recursive operations.

**Theorem 4.7.4 (Uniformization Theorem)** *Let $X, Y$ be sets over a natural structure $S$ and let $A \subseteq X \times Y$. If $Y$ is finally semi-recursive and $A$ semi-recursive over $S$, then there is a recursive operation $f :\subseteq X \rightrightarrows Y$ such that* $\mathrm{graph}(f) = A$.

We close this section with a characterization of effective subsets over metric structures.

**Theorem 4.7.5 (Recursive sets over metric structures)** *Let $X$ be a recursive metric space and $A \subseteq X$. Then $A$ is semi-recursive over $\mathbf{X}$, if and only if $A^c$ is co-r.e. closed. If, additionally, $X$ is complete and $A$ closed, then $A$ is finally semi-recursive over $\mathbf{X}$, if and only if $A$ is r.e. closed and $A$ is recursive over $\mathbf{X}$, if and only if $A$ is recursive.*

This result shows that the notions which we have introduced in Definition 4.7.1 are compatible with those used in computable analysis (see Appendix B for definitions). The details are left to the reader and can be found in [Bra99a].

## 4.8 The lower and upper structure of the real numbers

In the following sections we will see that several quasi-metric spaces give rise to perfect topological structures. In this section we start with two quasi-metric structures on the real numbers, the lower and the upper structure. As before, we denote by $\mathbb{R}_<$ and $\mathbb{R}_>$ the space of real numbers endowed with the lower and upper topology, respectively. For technical reasons we assume that $\mathbb{R}, \mathbb{R}_<$ and $\mathbb{R}_>$ are pairwise disjoint, but we will tacitly use the canonical bijections. In the following theorem we define the *lower structure* $\mathbf{R}_<$ and the *upper structure* $\mathbf{R}_>$ of the real numbers. For this purpose, we denote by "$x$" the identity $\mathrm{id}_{\mathbb{R}_<} : \mathbb{R}_< \to \mathbb{R}_<$ and by "$\inf_{n\in\mathbb{N}} y_n$" we denote the function

$$\mathrm{Inf} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}_>, (y_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} y_n.$$

Analogously, we define Sup (with sup instead of inf and $\mathbb{R}_<$ instead of $\mathbb{R}_>$). Following our convention, "$x < y$" denotes the semi-characteristic operation of the corresponding test:

$$c_{x<y} : \mathbb{R}_> \times \mathbb{R} \rightrightarrows \mathbb{N}, (x, y) \mapsto \begin{cases} \{0, 1\} & \text{if} \quad x < y \\ \{1\} & \text{else} \end{cases}$$

Analogously, we define "$x > y$" (with $\mathbb{R}_<$ instead of $\mathbb{R}_>$). It is easy to see that all given initial operations are lower semi-continuous and hence $\mathbf{R}_>$ and $\mathbf{R}_<$ are topological structures. Our goal is to prove the following theorem.

**Theorem 4.8.1 (The lower and upper structures of the reals)**
*The structures*

$$\mathbf{R}_> := \mathbf{R} \oplus \Big(\mathbb{R}_>; x, \inf\nolimits_{n\in\mathbb{N}} y_n, x < y\Big),$$
$$\mathbf{R}_< := \mathbf{R} \oplus \Big(\mathbb{R}_<; x, \sup\nolimits_{n\in\mathbb{N}} y_n, x > y\Big)$$

*are perfect topological structures. Moreover, the corresponding Dedekind representations yield standard representations of these structures.*

More precisely, $[\delta_{\mathbb{N}}, \delta_{\mathbb{R}}, \delta_{\mathbb{R}_<}]$ and $[\delta_{\mathbb{N}}, \delta_{\mathbb{R}}, \delta_{\mathbb{R}_>}]$ are standard representation of $\mathbf{R}_<$ and $\mathbf{R}_>$, respectively. As a direct consequence of this theorem and Theorem 4.5.9 we can characterize some recursive functions as semi-computable functions (cf. [WZ97, ZBW99]).

**Corollary 4.8.2** *The recursive functions* $f : \mathbb{R} \to \mathbb{R}_<$ *over* $\mathbf{R}_<$ *are the lower semi-computable functions, and the recursive functions* $f : \mathbb{R} \to \mathbb{R}_>$ *over* $\mathbf{R}_>$ *are the upper semi-computable functions.*

In particular, the recursive functions $f : \mathbb{R} \to \mathbb{R}_<$ over $\mathbf{R}_<$ are lower semi-continuous, and the recursive functions $f : \mathbb{R} \to \mathbb{R}_>$ over $\mathbf{R}_>$ are upper semi-continuous. Moreover, the recursive points in $\mathbf{R}_<$ and $\mathbf{R}_>$ are well-known (cf. [WZ98]).

**Corollary 4.8.3** *The recursive points $x \in \mathbb{R}_<$ over $\mathbf{R}_<$ are the left-computable and the recursive points $x \in \mathbb{R}_>$ over $\mathbf{R}_>$ are the right-computable points.*

Now it remains to prove Theorem 4.8.1. By definition the Dedekind representations $\delta_{\mathbb{R}_<} := \mathrm{Sup} \circ \delta_{\mathbb{R}}^{\infty}$ and $\delta_{\mathbb{R}_>} := \mathrm{Inf} \circ \delta_{\mathbb{R}}^{\infty}$ are recursive over $\mathbf{R}_<$ and $\mathbf{R}_>$, respectively. We have to prove that they also admit recursive right inverses.

**Proposition 4.8.4** *The operations* Sup *and* Inf *are recursive retractions over $\mathbf{R}_<$ and $\mathbf{R}_>$, respectively.*

**Proof.** We consider the case Inf. The operation Sup can be treated correspondingly. By definition Inf is recursive over $\mathbf{R}_>$. Moreover,

$$A := \{(x, n) \in \mathbb{R}_> \times \mathbb{N} : x < \alpha_{\mathbb{R}}(n)\}$$

is semi-recursive over $\mathbf{R}_>$ and thus by the Uniformization Theorem 4.7.4 there is a recursive operation $f : \mathbb{R}_> \rightrightarrows \mathbb{N}$ such that $\mathrm{graph}(f) = A$. By sequentialization we obtain a recursive operation $g : \mathbb{R}_> \rightrightarrows \mathbb{R}^{\mathbb{N}}$ over $\mathbf{R}_>$, defined by $g := \alpha_{\mathbb{R}}^{\mathbb{N}} \circ f^{\triangle}$, which is a right inverse of Inf, since

$$\mathrm{Inf} \circ g(x) = \inf\{\alpha_{\mathbb{R}}(n) : n \in \mathbb{N} \text{ and } x < \alpha_{\mathbb{R}}(n)\} = x$$

for all $x \in \mathbb{R}_>$. $\square$

It is worth mentioning that the previous proof is one of the rare places where we apply the sequentialization scheme. While metric structures have been proved to be perfect without any usage of the sequentialization scheme, it seems to be essential in the quasi-metric case. Since $\delta_{\mathbb{R}}$ is a recursive retraction over $\mathbf{R}$, so is $\delta_{\mathbb{R}}^{\infty}$ by Lemma 4.5.5. Consequently, we obtain the following corollary.

**Corollary 4.8.5** *The Dedekind representations $\delta_{\mathbb{R}_<}$ and $\delta_{\mathbb{R}_>}$ are recursive retractions over $\mathbf{R}_<$ and $\mathbf{R}_>$, respectively.*

It remains to prove that the structures $\mathbf{R}_<$ and $\mathbf{R}_>$ are effective with respect to $\delta_{\mathbb{R}_<}$ and $\delta_{\mathbb{R}_>}$, respectively. By definition, the operations $\mathrm{Sup}, \mathrm{Inf}$ are computable with respect to the corresponding Dedekind representations and the identity is computable too. We still have to investigate the tests $x > y$ and $x < y$. By $\hat{n}$ we denote the constant sequence $p \in \mathbb{N}^{\mathbb{N}}$ with $p(i) = n$ for all $i \in \mathbb{N}$.

**Proposition 4.8.6** *The semi-characteristic operations* $c_{x>y} : \mathbb{R}_< \times \mathbb{R} \rightrightarrows \mathbb{N}$ *and* $c_{x<y} : \mathbb{R}_> \times \mathbb{R} \rightrightarrows \mathbb{N}$ *are* $([\delta_{\mathbb{R}_<}, \delta_{\mathbb{R}}], \delta_{\mathbb{N}})$*– and* $([\delta_{\mathbb{R}_>}, \delta_{\mathbb{R}}], \delta_{\mathbb{N}})$*–computable, respectively.*

**Proof.** We consider the operation $c_{x<y}$. The other case can be proved correspondingly. We use the representation $\delta'_{\mathbb{R}_>}$ of $\mathbb{R}_>$, defined by $\delta'_{\mathbb{R}_>} := \mathrm{Inf} \circ \alpha_{\mathbb{R}}^{\mathbb{N}}$. It is easy to see that $\delta'_{\mathbb{R}_>} \equiv \delta_{\mathbb{R}_>}$. Define $F : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by

$$F\langle\langle p, r\rangle, q\rangle := \begin{cases} \hat{0} & \text{if } \quad \alpha_{\mathbb{R}} pq(0) < \alpha_{\mathbb{R}} rq(1) - 2^{-q(1)} \text{ and } q(2) = 0 \\ \hat{1} & \text{else} \end{cases}$$

for all $p, r, q \in \mathbb{N}^{\mathbb{N}}$. Then $F$ is computable since comparisons with respect to $\alpha_{\mathbb{R}}$ are decidable. Now $1 \in \{\delta_{\mathbb{N}} F\langle\langle p, r\rangle, q\rangle : q \in \mathbb{N}^{\mathbb{N}}\}$ and

$$\begin{aligned} 0 \in \{\delta_{\mathbb{N}} F\langle\langle p, r\rangle, q\rangle : q \in \mathbb{N}^{\mathbb{N}}\} &\iff (\exists n, k)\ \alpha_{\mathbb{R}} p(k) < \alpha_{\mathbb{R}} r(n) - 2^{-n} \\ &\iff \delta'_{\mathbb{R}_>}(p) < \delta_{\mathbb{R}}(r) \\ &\iff 0 \in c_{x<y}(\delta'_{\mathbb{R}_>}(p), \delta_{\mathbb{R}}(r)), \end{aligned}$$

for all $p \in \mathrm{dom}(\delta'_{\mathbb{R}_>})$ and $r \in \mathrm{dom}(\delta_{\mathbb{R}})$, i.e. $c_{x<y}$ is $([\delta'_{\mathbb{R}_>}, \delta_{\mathbb{R}}], \delta_{\mathbb{N}})$–computable via $F$. □

This finishes the proof of Theorem 4.8.1. The theorem does not state that $\mathbf{R}_<$, $\mathbf{R}_>$ are *strongly* perfect or complete structures. This can be achieved if we use the extended real lines $\widehat{\mathbb{R}}_< := \mathbb{R}_< \cup \{\infty\}$ and $\widehat{\mathbb{R}}_> := \mathbb{R}_> \cup \{-\infty\}$ with the canonical extension of the order. In this case $\mathrm{Inf} : \mathbb{R}^{\mathbb{N}} \to \widehat{\mathbb{R}}_>$ and $\mathrm{Sup} : \mathbb{R}^{\mathbb{N}} \to \widehat{\mathbb{R}}_<$ are obviously total. If we define the *extended lower and upper structures* $\widehat{\mathbf{R}}_<, \widehat{\mathbf{R}}_>$ analogously to $\mathbf{R}_<$, $\mathbf{R}_>$, then we obtain the following result.

**Theorem 4.8.7 (The extended structures of the reals)** *The extended structures* $\widehat{\mathbf{R}}_<$ *and* $\widehat{\mathbf{R}}_>$ *are strongly perfect complete topological structures.*

The proof is completely analogous to the proof of Theorem 4.8.1.

## 4.9 Quasi-metric structures

In this section we want to prove a theorem for strong semi-recursive quasi-metric spaces which corresponds to Theorem 4.6.7 for recursive metric spaces. If $(X, Y, d)$ is an upper generated quasi-metric space, then we will denote by $X_>$ the space $X$ endowed with the weak upper topology of $(X, Y, d)$.

**Theorem 4.9.1 (Quasi-metric structures)** *If $(X, Y, d)$ is a strong semi-recursive quasi-metric space, then*

$$\mathbf{X}_> := \mathbf{Y} \oplus \mathbf{R}_> \oplus (X_>; \mathrm{id}_{X_>}, \mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X_>, d : X_> \times Y \to \mathbb{R}_>)$$

*is a perfect topological structure. The corresponding Dedekind representation yields a standard representation of this structure.*

**Proof.** Let $(X, Y, d)$ be a strong semi-recursive quasi-metric space. By Theorem 4.6.7 $\mathbf{Y}$ is a perfect topological structure and by Theorem 4.8.1 $\mathbf{R}_>$ is a perfect topological structure. Thus, by Theorem 4.5.12 $\mathbf{Y} \oplus \mathbf{R}_>$ is a perfect topological structure as well. By Lemma 2.2.4 (1) we obtain that $d : X_> \times Y \to \mathbb{R}_>$ is continuous and by Corollary 3.3.3 it follows that $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X_>$ is continuous. Altogether, $\mathbf{X}_>$ is actually a topological structure. Now let $\delta_{X_>}$ be the Dedekind representation of $X_>$ and consider the corresponding standard representation $\delta_{\mathbf{X}_>} := [\delta_{\mathbb{N}}, \delta_{\mathbb{R}}, \delta_Y, \delta_{\mathbb{R}_>}, \delta_{X_>}]$ of $\mathbf{X}_>$. By the first mentioned two Theorems and Proposition 3.2.4 we obtain that $\mathbf{X}_>$ is effective via $\delta_{\mathbf{X}_>}$. Moreover, $\delta_{X_>} = \mathrm{Inf} \circ \delta_Y^\infty$ is obviously recursive over $\mathbf{X}_>$. Since $\delta_Y$ is a recursive retraction over $\mathbf{Y}$, so is $\delta_Y^\infty$ by Lemma 4.5.5. In order to complete the proof, it suffices to show that Inf also admits a recursive right inverse over $\mathbf{X}_>$. This is done in the following proposition. □

As mentioned in the proof, $\delta_{\mathbf{X}_>} := [\delta_{\mathbb{N}}, \delta_{\mathbb{R}}, \delta_Y, \delta_{\mathbb{R}_>}, \delta_{X_>}]$ is a standard representation of the structure $\mathbf{X}_>$. The most technical part of the proof of the previous theorem is included in the proof of the following proposition which is a kind of a structural version of the proof of Proposition 3.6.1. Moreover, parts of the proof are even postponed to Appendix A.

**Proposition 4.9.2** *If $(X, Y, d)$ is a strong semi-recursive quasi-metric space, then* $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X_>$ *is a recursive retraction over* $\mathbf{X}_>$.

**Proof.** Let $(X, Y, d, \alpha)$ be a strong semi-recursive quasi-metric space and let $Y \subseteq X$ be strongly dense in this space with constant $c \in \mathbb{N}$. The sets

$$A := \left\{(x, \langle l, m\rangle) \in X_> \times \mathbb{N} : d(x, \alpha(m)) < \frac{1}{c} \cdot 2^{-l-2}\right\}$$

and

$$\begin{aligned} B \;:=\; & \Big\{(x, k, \langle l, m\rangle, n) \in X_> \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \\ & \quad d_\star(\alpha(n), \alpha(m)) < 2^{-k-l-1} \text{ and } d(x, \alpha(n)) < \frac{1}{c} \cdot 2^{-k-l-3}\Big\} \end{aligned}$$

are semi-recursive over $\mathbf{X}_>$. Since $\mathbf{X}_>$ is a natural structure, by the Uniformization Theorem 4.7.4 there are recursive operations $f :\subseteq X_> \rightrightarrows \mathbb{N}$, $g :\subseteq X_> \times \mathbb{N} \times \mathbb{N} \rightrightarrows \mathbb{N}$ over $\mathbf{X}_>$ such that $\text{graph}(f) = A$ and $\text{graph}(g) = B$. Then $h :\subseteq X_> \times \mathbb{N} \times \mathbb{N} \rightrightarrows X_> \times \mathbb{N} \times \mathbb{N}$, defined by

$$h(x, k, \langle l, m\rangle) := \Big(x, k+1, \Big\langle l, g(x, k, \langle l, m\rangle)\Big\rangle\Big)$$

is recursive over $\mathbf{X}_>$ too. Now we use the projection $\pi_2 : \mathbb{N} \to \mathbb{N}, \langle l, m\rangle \mapsto m$, the projection $\text{pr}_3 : X_> \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and the sequential iteration scheme, as given by Definition A.1, in order to define a function $G :\subseteq X_> \times \mathbb{N} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ by $G(x, \langle l, m\rangle) := \pi_2^{\mathbb{N}} \circ \text{pr}_3^{\mathbb{N}} \circ h^{\nabla}(x, 0, \langle l, m\rangle)$. By Proposition A.3 it follows that $G$ is recursive over $\mathbf{X}_>$ as well. By definition of $G$ one obtains

$$\begin{aligned}
& G(x, \langle l, m\rangle) \\
= \; & \{(m_k)_{k\in\mathbb{N}} : m = m_0 \text{ and } (\forall k)(x, k, \langle l, m_k\rangle, m_{k+1}) \in B\} \\
= \; & \Big\{(m_k)_{k\in\mathbb{N}} : m = m_0 \text{ and } (\forall k)\Big(\; d_\star(\alpha(m_{k+1}), \alpha(m_k)) < 2^{-k-l-1} \text{ and} \\
& \qquad\qquad\qquad d(x, \alpha(m_{k+1})) < \frac{1}{c} \cdot 2^{-k-l-3}\Big)\Big\}
\end{aligned}$$

for all $(x, \langle l, m\rangle) \in A$. Similarly as in the proof of Proposition 3.6.1 one can show that there actually exists some sequence $(m_k)_{k\in\mathbb{N}} \in G(x, \langle l, m\rangle)$ for any $(x, \langle l, m\rangle) \in A$, which in particular proves $A \subseteq \text{dom}(G)$. This follows from the fact that $\alpha$ is dense in $(Y, d_\star)$ and $Y$ is strongly dense in $(X, d)$ with constant $c$. Now we define $H :\subseteq X_> \times \mathbb{N} \rightrightarrows Y$ by $H := \text{Lim} \circ \alpha^{\mathbb{N}} \circ G$ and $F : X_> \rightrightarrows Y^{\mathbb{N}}$ by $F = H^{\mathbb{N}} \circ \Pi \circ (S, f^{\triangle})$, where $S : X_> \to X_>^{\mathbb{N}}$ maps any $x \in X_>$ to the constant sequence with value $x$ and $\Pi : X_>^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to (X_> \times \mathbb{N})^{\mathbb{N}}$ is the canonical bijection. Then $H$ and $F$ are recursive over $\mathbf{X}_>$ and again similarly as in the proof of Proposition 3.6.1, it follows that $\{y_n : n \in \mathbb{N}\}$ is dense in $\{y \in Y : x \sqsubseteq y\}$ for each $x \in X_>$ and $(y_n)_{n\in\mathbb{N}} \in F(x)$. By assumption $(X, Y, d)$ is upper generated, hence by Lemma 2.4.2 it follows that $\text{Inf} \circ F = \text{id}_{X_>}$, i.e. $F$ is a right inverse of Inf. This completes the proof of the theorem. □

The reader should notice that the proof essentially uses the sequential iteration scheme and Proposition A.3. Alternatively, one could add the sequential iteration scheme to the recursive closure schemes.

If $\text{Inf} :\subseteq Y^{\mathbb{N}} \to X_>$ is total, then we could even obtain a strongly perfect and complete extended structure $\widehat{\mathbf{X}}_>$ by substituting $\widehat{\mathbf{R}}_>$ for $\mathbf{R}_>$. By Theorem 4.8.7 we obtain the following result, analogously to Theorem 4.9.1 (the proof is based on the additional fact that $Y$ is complete and thus $\mathbf{Y}$ is a strongly perfect complete structure by Theorem 4.6.7).

**Theorem 4.9.3 (Extended quasi-metric structures)** *If* $(X, Y, d)$ *is a strong semi-recursive quasi-metric space and* $\mathrm{Inf} :\subseteq Y^{\mathbb{N}} \to X_>$ *is total, then*

$$\widehat{\mathbf{X}}_> := \mathbf{Y} \oplus \widehat{\mathbf{R}}_> \oplus (X_>; \mathrm{id}_{X_>}, \mathrm{Inf} : Y^{\mathbb{N}} \to X_>, d : X_> \times Y \to \widehat{\mathbb{R}}_>)$$

*is a strongly perfect complete topological structure.*

In the following sections we will derive some straightforward conclusions from the results of this section. In particular, we will apply Theorem 4.9.1 to the hyper and function spaces investigated in the previous chapters.

## 4.10 The lower and upper structures of compact subsets

In this section we want to endow the set $\mathcal{K}(X)$ of non-empty compact subsets of complete recursive metric spaces $(X, d)$ with a quasi-metric structure. By $\mathcal{K}_<(X)$ and $\mathcal{K}_>(X)$ we denote the hyperspace endowed with the lower and upper Vietoris topology, respectively. In the following corollary we will define the *lower structure of compact subsets* $\mathcal{K}_<(\mathbf{X})$ and the *upper structure of compact subsets* $\mathcal{K}_>(\mathbf{X})$ For this purpose, we denote by "$A$" the identity $\mathrm{id}_{\mathcal{K}_<} : \mathcal{K}_<(X) \to \mathcal{K}_<(X)$, by "$\overline{\bigcup_{n=0}^{\infty} B_n}$" the function

$$\mathrm{Sup} :\subseteq \mathcal{K}(X)^{\mathbb{N}} \to \mathcal{K}_<(X), (B_n)_{n\in\mathbb{N}} \mapsto \overline{\bigcup_{n=0}^{\infty} B_n}$$

and by "$\overline{d'_{\mathcal{K}}}(A, B)$" the conjugate quasi-metric

$$\overline{d'_{\mathcal{K}}} : \mathcal{K}_<(X) \times \mathcal{K}(X) \to \mathbb{R}_>, (A, B) \mapsto \sup_{b\in B} \inf_{a\in A} d(a, b).$$

Analogously, we interpret the operations of $\mathcal{K}_>(\mathbf{X})$. As a corollary of Theorem 4.9.1 and Corollary 3.7.5 we obtain the following result.

**Corollary 4.10.1 (Lower and upper structures of compact subsets)** *If* $X$ *is a complete recursive metric space, then*

$$\mathcal{K}_<(\mathbf{X}) := \mathcal{K}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \left(\mathcal{K}_<(X); A, \overline{\textstyle\bigcup_{n=0}^{\infty} B_n}, \overline{d'_{\mathcal{K}}}(A, B)\right),$$
$$\mathcal{K}_>(\mathbf{X}) := \mathcal{K}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \left(\mathcal{K}_>(X); A, \textstyle\bigcap_{n=0}^{\infty} B_n, d'_{\mathcal{K}}(A, B)\right)$$

*are perfect topological structures.*

In case that $X$ is compact, we can easily extend $\mathcal{K}_<(\mathbf{X})$ to a *strongly* perfect complete topological structure. As a consequence of Theorem 4.9.3 we obtain the following corollary.

**Corollary 4.10.2 (Extended lower structure of compact subsets)** *If $X$ is a compact recursive metric space, then*

$$\widehat{\mathcal{K}}_<(\mathbf{X}) := \mathcal{K}(\mathbf{X}) \oplus \widehat{\mathbf{R}}_> \oplus \left(\mathcal{K}_<(X); A, \overline{\bigcup_{n=0}^{\infty} B_n}, \overline{d'_{\mathcal{K}}}(A,B)\right)$$

*is a strongly perfect complete topological structure.*

The extension of $\mathcal{K}_>(\mathbf{X})$ to a strongly perfect complete structure would require further considerations. In this case $\mathbf{R}_>$ would have to be replaced by $\widehat{\mathbf{R}}_>$, $\mathcal{K}_>(X)$ would have to be replaced by $\widehat{\mathcal{K}}_>(X) := \mathcal{K}_>(X) \cup \{\emptyset\}$ and the quasi-metric $d'_{\mathcal{K}}$ would have to be extended correspondingly. We leave such an extension to the reader.

## 4.11 The upper structures of semi-continuous functions

In this section we want to endow the sets $\mathcal{LSC}(X)$ and $\mathcal{USC}(X)$ of lower and upper semi-continuous functions $f : X \to \mathbb{R}$, respectively, with a quasi-metric structure. We assume that both spaces are endowed with their corresponding weak upper topology which by Proposition 2.6.3 is the lower compact open topology and the upper compact open topology, respectively. In the following corollary we will define structures $\mathcal{USC}(\mathbf{X})$ and $\mathcal{LSC}(\mathbf{X})$ for recursively locally compact recursive metric spaces $X$. In case of $\mathcal{USC}(\mathbf{X})$ we denote for this purpose by "$f$" the identity $\mathrm{id}_{\mathcal{USC}(X)}$, by "$\inf_{n\in\mathbb{N}} g_n$" the function

$$\mathrm{Inf} :\subseteq \mathcal{C}(X)^{\mathbb{N}} \to \mathcal{USC}(X), (g_n)_{n\in\mathbb{N}} \mapsto \inf_{n\in\mathbb{N}} g_n$$

and by "$d_{\mathcal{USC}}(f,g)$" the function

$$d_{\mathcal{USC}} : \mathcal{USC}(X) \times \mathcal{C}(X) \to \mathbb{R}_>, (f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{|f \dot{-} g|_{K_i}}{1 + |f \dot{-} g|_{K_i}},$$

where we assume that $(K_i)_{i\in\mathbb{N}}$ is some fixed recursive exhausting sequence of $X$. Analogously, we interpret the operations of $\mathcal{LSC}(\mathbf{X})$. As a corollary of Theorem 4.9.1 and Theorem 3.8.2 we obtain the following result.

**Corollary 4.11.1 (Structure of semi-continuous functions)** *If $X$ is a recursively locally compact recursive metric space, then*

$$\mathcal{USC}(\mathbf{X}) := \mathcal{C}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \Big(\mathcal{USC}(X); f, \inf_{n\in\mathbb{N}} g_n, d_{\mathcal{USC}}(f,g)\Big),$$
$$\mathcal{LSC}(\mathbf{X}) := \mathcal{C}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \Big(\mathcal{LSC}(X); f, \sup_{n\in\mathbb{N}} g_n, d_{\mathcal{LSC}}(f,g)\Big)$$

*are perfect topological structures.*

We omit a discussion of corresponding strongly perfect complete structures (in this case one had to extend the function spaces to functions $f : X \to \widehat{\mathbb{R}}_<$ and $f : X \to \widehat{\mathbb{R}}_>$, respectively). The recursive operations over the structures $\mathcal{USC}(\mathbf{X})$ and $\mathcal{LSC}(\mathbf{X})$ are by Theorem 4.5.9 the corresponding computable operations. We just mention the special case of recursive points (as a corollary of Theorem 5.1.2 which we will prove independently in Chapter 5).

**Corollary 4.11.2** *Let $X$ be a recursively locally compact recursive metric space. The recursive points $f \in \mathcal{USC}(X)$ over $\mathcal{USC}(\mathbf{X})$ and $f \in \mathcal{LSC}(X)$ over $\mathcal{LSC}(\mathbf{X})$ are the computable functions $f : X \to \mathbb{R}_>$ and $f : X \to \mathbb{R}_<$ (thus, the upper and lower semi-computable functions), respectively.*

## 4.12 The lower and upper structures of closed subsets

In this section we want to endow the set $\mathcal{A}(X)$ of non-empty closed subsets of nice recursively locally compact recursive metric spaces $(X, d)$ with a quasi-metric structure. By $\mathcal{A}_<(X)$ and $\mathcal{A}_>(X)$ we denote the hyperspace endowed with the lower and upper Fell topology, respectively. In the following corollary we will define structures $\mathcal{A}_<(\mathbf{X})$ and $\mathcal{A}_>(\mathbf{X})$. For this purpose we denote by "$A$" the identity $\mathrm{id}_{\mathcal{A}_>} : \mathcal{A}_>(X) \to \mathcal{A}_>(X)$, by "$\bigcap_{n=0}^\infty B_n$" the function

$$\mathrm{Inf} :\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}_>(X), (B_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} B_n$$

and by "$d'_{\mathcal{A}}(A,B)$" the function

$$d'_{\mathcal{A}} : \mathcal{A}_>(X) \times \mathcal{A}(X) \to \mathbb{R}_>, (A,B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} |d_B \dot{-} d_A|_{K_i},$$

where we assume that $(K_i)_{i\in\mathbb{N}}$ is some fixed recursive exhausting sequence of $X$ such that $d$ is nice with respect to $(K_i)_{i\in\mathbb{N}}$. Analogously, we interpret the operations of $\mathcal{A}_<(\mathbf{X})$. As a corollary of Theorem 4.9.1 and Corollary 3.9.5 we obtain the following result.

**Corollary 4.12.1 (Lower and upper structures of closed subsets)** *If $X$ is a nice recursively locally compact recursive metric space, then*

$$\mathcal{A}_<(\mathbf{X}) := \mathcal{A}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \left(\mathcal{A}_<(X); A, \overline{\bigcup_{n=0}^\infty B_n}, \overline{d'_\mathcal{A}}(A,B)\right),$$
$$\mathcal{A}_>(\mathbf{X}) := \mathcal{A}(\mathbf{X}) \oplus \mathbf{R}_> \oplus \left(\mathcal{A}_>(X); A, \bigcap_{n=0}^\infty B_n, d'_\mathcal{A}(A,B)\right)$$

*are perfect topological structures.*

The recursive operations over these structures are by Theorem 4.5.9 the corresponding computable operations. We just mention the special case of recursive points (as a corollary of Theorem 5.1.8 which we will prove independently in Chapter 5).

**Corollary 4.12.2** *Let $X$ be a nice recursively locally compact recursive metric space The recursive points $A \in \mathcal{A}_<(X)$ over $\mathcal{A}_<(\mathbf{X})$ and $A \in \mathcal{A}_>(X)$ over $\mathcal{A}_>(\mathbf{X})$ are the r.e. and co-r.e. closed subsets of $X$, respectively.*

Since the function $\mathrm{Sup} :\subseteq \mathcal{A}(X)^\mathbb{N} \to \mathcal{A}_>(X)$ is even total, we easily obtain an extended strongly perfect and complete topological structure under the same assumptions as in Corollary 4.12.1.

**Corollary 4.12.3 (Extended lower structure of closed subsets)** *If $X$ is a nice recursively locally compact recursive metric space, then*

$$\widehat{\mathcal{A}}_<(\mathbf{X}) := \mathcal{A}(\mathbf{X}) \oplus \widehat{\mathbf{R}}_> \oplus \left(\mathcal{A}_<(X); A, \overline{\bigcup_{n=0}^\infty B_n}, \overline{d'_\mathcal{A}}(A,B)\right)$$

*is a strongly perfect complete topological structure.*

The quasi-metric structures of this chapter are collected together with the corresponding metric structures in the Tables of Perfect Structures in the Appendix.

# Chapter 5

# Miscellany

In this chapter we have compiled a number of miscellaneous results. In the first section we characterize some Dedekind representations of quasi-metric spaces. These characterizations finally prove that the theory, developed in the previous chapters, is very well embedded into computable analysis. In the second section we briefly mention some computable embeddings and injections which naturally connect some of the hyper and function spaces which we have investigated. Some of these embeddings and injections have already been used implicitly. In the third section we discuss the possibility of effective quasi-metrizability. In particular, we study a canonical construction of a quasi-metric for second-countable $T_0$–spaces.

## 5.1 Characterization of some Dedekind representations

The purpose of this section is to connect the results of the previous chapters with well-known facts of the representations based approach to computable analysis [Wei00]. Especially, we will prove that our quasi-metric function space representations are equivalent to function space representations studied in [WZ00] (for the Euclidean case) and that the quasi-metric hyperspace representations (of closed subsets) are equivalent to certain hyper space representations studied in [BPar] (and in [BW99] for the Euclidean case).

We first mention that our representations[1] of the reals are obviously equiv-

[1] We recall that in the cited references representations are considered as surjective mappings $\delta :\subseteq \Sigma^\omega \to X$ defined on the set $\Sigma^\omega$ of infinite sequences over some finite alphabet $\Sigma$. In contrast to that we consider representations as surjective mappings $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$. However, we tacitly ignore this difference since it does not cause any problems for our applications.

alent to those used in the references, i.e. using the notations from [Wei00] we obtain $\delta_{\mathbb{R}} \equiv \rho$, $\delta_{\mathbb{R}_<} \equiv \rho_<$ and $\delta_{\mathbb{R}_>} \equiv \rho_>$. Now we recall that for any two representations $\delta_X$ of $X$ and $\delta_Y$ of $Y$ there exists a canonical representation $[\delta_X \to \delta_Y]$ of the set $\mathcal{C}(\delta_X, \delta_Y)$ of $(\delta_X, \delta_Y)$–continuous functions $f : X \to Y$ (see [Wei00, Bra02a]). This function space representation allows *evaluation* and *type conversion* (also called *Curry operation*) and actually it can be characterized by these properties.

**Proposition 5.1.1 (Evaluation and type conversion)** *Let* $(X, \delta_X)$ *and* $(Y, \delta_Y)$ *be represented spaces and let* $\delta$ *be a representation of* $\mathcal{C}(\delta_X, \delta_Y)$. *Then*

*(1)* $\delta \leq [\delta_X \to \delta_Y]$, *if and only if* $\mathrm{ev} : \mathcal{C}(\delta_X, \delta_Y) \times X \to Y, (f, x) \mapsto f(x)$ *is* $([\delta, \delta_X], \delta_Y)$*–computable,*

*(2)* $[\delta_X \to \delta_Y] \leq \delta$, *if and only if for any represented space* $(Z, \delta_Z)$ *and any function* $f : Z \times X \to Y$ *which is* $([\delta_Z, \delta_X], \delta_Y)$*–computable, it follows that the transposed function* $\check{f} : Z \to \mathcal{C}(\delta_X, \delta_Y)$, *defined by* $\check{f}(z)(x) := f(z, x)$, *is* $(\delta_Z, \delta)$*–computable.*

**Proof.** The proof of (1) follows directly from Lemma 3.3.14 of [Wei00], the only if part of (2) follows from Theorem 3.3.15 of [Wei00] and the if part of (2) can be obtain from (1) and an application of the premise to the space $(Z, \delta_Z) = (\mathcal{C}(\delta_X, \delta_Y), [\delta_X \to \delta_Y])$ and the evaluation function $f = \mathrm{ev}$ (such that $\check{f} = \check{\mathrm{ev}} = \mathrm{id}_{\mathcal{C}(\delta_X, \delta_Y)}$ induces the reduction). □

We mention that the proof shows that for the if part of (2) it suffices to formulate (2) just for the case of $Z = \mathcal{C}(\delta_X, \delta_Y)$. Using the previously mentioned properties, we can prove the following theorem.

**Theorem 5.1.2 (Function space representations)** *If* $X$ *is a recursively locally compact recursive metric space, then*

*(1)* $\delta_{\mathcal{C}(X)} \equiv [\delta_X \to \delta_{\mathbb{R}}]$,

*(2)* $\delta_{\mathcal{USC}(X)} \equiv [\delta_X \to \delta_{\mathbb{R}_>}]$,

*(3)* $\delta_{\mathcal{LSC}(X)} \equiv [\delta_X \to \delta_{\mathbb{R}_<}]$.

**Proof.**

(1) By Proposition 5.1.1 it suffices to prove that $\delta_{\mathcal{C}(X)}$ allows evaluation and type conversion. The statement concerning evaluation follows directly from Corollary 4.4.56 of [Bra99a]. The statement concerning type conversion is stated in Theorem 4.4.55 of [Bra99a] for separable metric spaces

$Z$. But this suffices in order to conclude $[\delta_X \to \delta_{\mathbb{R}}] \leq \delta_{\mathcal{C}(X)}$ since by the proof of Proposition 5.1.1 we have to apply transposition just in case $Z = \mathcal{C}(\delta_X, \delta_{\mathbb{R}}) = \mathcal{C}(X)$, which is a separable metric space in case that $X$ is a recursively locally compact recursive metric space.

(2) "$\leq$" By Proposition 5.1.1 it suffices to prove that the evaluation function $\mathrm{ev} : \mathcal{C}(\delta_X, \delta_{\mathbb{R}_>}) \times X \to \mathbb{R}_>$ is $([\delta_{\mathcal{USC}(X)}, \delta_X], \delta_{\mathbb{R}_>})$–computable. But this follows since $\delta_{\mathcal{USC}(X)} = \mathrm{Inf} \circ \delta_{\mathcal{C}(X)}$, $\mathrm{ev}(\inf_{i\in\mathbb{N}} f_i, x) = \inf_{i\in\mathbb{N}} \mathrm{ev}(f_i, x)$ for all $f_i \in \mathcal{C}(X)$, $x \in X$, $i \in \mathbb{N}$, $\mathrm{ev} : \mathcal{C}(X) \times X \to \mathbb{R}$ is $([\delta_{\mathcal{C}(X)}, \delta_X], \delta_{\mathbb{R}})$–computable by (1) and $\mathrm{Inf} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}_>$ is $(\delta_{\mathbb{R}}^{\infty}, \delta_{\mathbb{R}_>})$–computable.
"$\geq$" By Proposition 3.2.4 it suffices to prove that

$$d_{\mathcal{USC}} : \mathcal{USC}(X) \times \mathcal{C}(X) \to \mathbb{R}_>, (f, g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{|f \dot{-} g|_{K_i}}{1 + |f \dot{-} g|_{K_i}}$$

is $([[\delta_X \to \delta_{\mathbb{R}_>}], \delta_{\mathcal{C}(X)}], \delta_{\mathbb{R}_>})$–computable. But this is a consequence of the fact that $(x_i)_{i\in\mathbb{N}} \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{x_i}{1+x_i}$ is $(\delta_{\mathbb{R}_>}^{\mathbb{N}}, \delta_{\mathbb{R}_>})$–computable (for sequences of non-negative reals) and of the following Lemma 5.1.3 and Proposition 5.1.4 which will be proved independently.

(3) Can be deduced from (2) since $f \mapsto -f$ is $(\delta_{\mathcal{LSC}(X)}, \delta_{\mathcal{USC}(X)})$–computable and $([\delta_X \to \delta_{\mathbb{R}_>}], [\delta_X \to \delta_{\mathbb{R}_<}])$–computable.

□

This result proves that our Dedekind representations of the function spaces $\mathcal{LSC}(X)$ and $\mathcal{USC}(X)$ are equivalent to naturally defined and well-known representations from computable analysis (see [WZ00] for the Euclidean case). Moreover, this finally proves Proposition 3.8.3, which is a direct corollary of Theorem 5.1.2. The proof of the previous theorem is completed by the following two computability results on functions. The first result shows that the arithmetic difference is computable on function spaces in a certain sense.

**Lemma 5.1.3** *Let $X$ be a recursively locally compact recursive metric space. Then*

$$\dot{-} : \mathcal{USC}(X) \times \mathcal{C}(X) \to \mathcal{USC}(X), (f, g) \mapsto f \dot{-} g$$

*is $([[\delta_X \to \delta_{\mathbb{R}_>}], \delta_{\mathcal{C}(X)}], [\delta_X \to \delta_{\mathbb{R}_>}])$–computable.*

**Proof.** This is an easy application of Proposition 5.1.1, of the characterization $\delta_{\mathcal{C}(X)} \equiv [\delta_X \to \delta_{\mathbb{R}}]$ (which holds by Theorem 5.1.2 (1)) and the fact that $\dot{-} : \mathbb{R}_> \times \mathbb{R} \to \mathbb{R}_>, (x, y) \mapsto x \dot{-} y$ is $([\delta_{\mathbb{R}_>}, \delta_{\mathbb{R}}], \delta_{\mathbb{R}_>})$–computable. □

The next result on functions states that on compact subsets the supremum of upper semi-continuous functions can be effectively approximated from above. We mention that obviously $\delta_{\mathcal{K}(X)} \equiv \delta_{\text{Hausdorff}} \equiv \delta_{\text{min-cover}}$ holds with the representations $\delta_{\text{Hausdorff}}$ and $\delta_{\text{min-cover}}$ defined as in [BPar]. The first equivalence holds since both representations are defined almost identical[2] and the second equivalence is a consequence of Theorem 4.12 of [BPar].

**Proposition 5.1.4** *Let $X$ be a recursively locally compact recursive metric space. Then*

$$M : \mathcal{USC}(X) \times \mathcal{K}(X) \to \mathbb{R}_>, (f, K) \mapsto \sup_{x \in K} f(x)$$

*is $([[\delta_X \to \delta_{\mathbb{R}_>}], \delta_{\mathcal{K}(X)}], \delta_{\mathbb{R}_>})$–computable.*

**Proof.** We sketch the proof which is based on results from [Wei01] (or [Wei02]). First of all, we tacitly assume $\Sigma^\omega \subseteq \mathbb{N}^\mathbb{N}$ for this proof and we recall that with the notations from [Wei01] and the results from [BPar] we obtain $\delta_{\mathcal{K}(Y)} \equiv \delta_{\text{min-cover}} \equiv \kappa_{\text{mc}}$ for all computable metric spaces $Y$ and $\delta_{\mathcal{K}(\Sigma^\omega)} \equiv \delta_{\mathcal{K}}^{>} \equiv \kappa_{>}^{\Sigma^\omega}$. By Theorem 4.4 of [Wei01] there is a representation $\delta$ of $X$ such that $\delta \equiv \delta_X$ and $\mathcal{K}(X) \to \mathcal{K}(\Sigma^\omega), K \mapsto \delta^{-1}(K)$ is $(\delta_{\mathcal{K}(X)}, \delta_{\mathcal{K}(\Sigma^\omega)})$–computable. Then $[\delta_X \to \delta_{\mathbb{R}_>}] \equiv [\delta \to \delta_{\mathbb{R}_>}]$. Now, by definition, $[\delta \to \delta_{\mathbb{R}_>}](p) = f$, if and only if $\delta_{\mathbb{R}_>}\eta_p(q) = f\delta(q)$ for all $q \in \text{dom}(f\delta)$ (where $\eta$ is some standard representation of the set of continuous functions $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with $G_\delta$–domain). By Theorem 3.3 of [Wei01] we obtain that the mapping $T : (F, K) \mapsto F(K)$ is $([\eta, \delta_{\mathcal{K}(\Sigma^\omega)}], \delta_{\mathcal{K}(\mathbb{N}^\mathbb{N})})$–computable for all compact $K \subseteq \text{dom}(f\delta)$. It is easy to see that $S :\subseteq \mathcal{K}(\mathbb{N}^\mathbb{N}) \to \mathbb{R}_>, K \mapsto \sup \delta_{\mathbb{R}_>}(K)$ is $(\delta_{\mathcal{K}(\mathbb{N}^\mathbb{N})}, \delta_{\mathbb{R}_>})$–computable for all compact $K \subseteq \text{dom}(\delta_{\mathbb{R}_>})$. Altogether, we obtain

$$ST(F, \delta^{-1}(K)) = \sup \delta_{\mathbb{R}_>} F(\delta^{-1}(K)) = \sup f\delta(\delta^{-1}(K)) = \sup f(K) = M(f, K)$$

for $[\delta \to \delta_{\mathbb{R}_>}](p) = f$ and $\eta_p = F$. Hence, $M$ is $([[\delta_X \to \delta_{\mathbb{R}_>}], \delta_{\mathcal{K}(X)}], \delta_{\mathbb{R}_>})$–computable. □

As a final goal of this section we want to prove a theorem for the hyperspace $\mathcal{A}(X)$ which corresponds to the previous theorem for function spaces. In order to do this, we first have to compare different notions of "effective local compactness" which have been used in this work and in [BPar]. The following lemma shows that any nice recursively locally compact recursive metric space has *nice closed balls* (1) and fulfills the *effective covering property* (2) in the sense of [BPar]. By $\overline{B}(x, \varepsilon) := \{y \in X : d(x, y) \leq \varepsilon\}$ we denote the *closed balls*

[2] In contrast to [BPar], we tacitly consider all these representations as representations of the hyperspace of *non-empty* compact subsets of $X$.

of a metric space $(X, d)$. We recall that in general $\overline{B(x,\varepsilon)} \subseteq \overline{B}(x,\varepsilon)$, while "$\supseteq$" only holds in special cases.

**Lemma 5.1.5** *Let $(X, d, \alpha)$ be a nice recursively locally compact recursive metric space. Then:*

*(1) $\overline{B}(x,\varepsilon)$ is compact or $\overline{B}(x,\varepsilon) = X$ for all $x \in X$ and $\varepsilon > 0$,*

*(2) $\{\langle\langle n,r\rangle, \langle n_0,r_0\rangle, ..., \langle n_k,r_k\rangle, k\rangle : \overline{B}(\alpha(n),\overline{r}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i),\overline{r_i})\}$ is r.e.*

**Proof.** Let $(X, d, \alpha)$ be a nice recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

(1) Let $x \in X$ and $\varepsilon > 0$. Since $d$ is bounded by 1, it follows $\overline{B}(x,\varepsilon) = X$ if $\varepsilon \geq 1$. Thus, let us assume $\varepsilon < 1$. Then $y \in \overline{B}(x,\varepsilon)$ implies $d(x,y) \leq \varepsilon < 1$. Let $i \in \mathbb{N}$ be such that $x \in K_i$, then $y \in K_{i+1}^{\circ}$ follows since $d$ is nice with respect to $(K_i)_{i\in\mathbb{N}}$ and consequently $\overline{B}(x,\varepsilon) \subseteq K_{i+1}$. Thus, $\overline{B}(x,\varepsilon)$ is a closed subset of a compact set and hence compact itself.

(2) Similarly as in (1) we first prove (*): $B(x,\varepsilon)$ is relatively compact for any $\varepsilon \in (0,1]$ and $B(x,\varepsilon) = X$ for all $\varepsilon > 1$ and $x \in X$. Let $\varepsilon \in (0,1]$ and $x \in X$. Then $y \in B(x,\varepsilon)$ implies $d(x,y) < \varepsilon \leq 1$. Let $i \in \mathbb{N}$ be such that $x \in K_i$, then $y \in K_{i+1}^{\circ}$ follows since $d$ is nice and consequently $B(x,\varepsilon) \subseteq K_{i+1}$ is relatively compact. Now let $\varepsilon > 1$. Then $B(x,\varepsilon) = X$ since $d$ is bounded by 1. This proves (*).
Now let $\langle\langle n,r\rangle, \langle n_0,r_0\rangle, ..., \langle n_k,r_k\rangle, k\rangle \in \mathbb{N}$. If $X$ is not compact and $\overline{r} \geq 1$, then $\overline{B}(\alpha(n),\overline{r}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i),\overline{r_i})$ implies that there is an $i \in \{0,...,k\}$ such that $\overline{r_i} > 1$ since otherwise all $B(\alpha(n_i),\overline{r_i})$ are relatively compact by (*) and thus $\bigcup_{i=0}^{k} B(\alpha(n_i),\overline{r_i})$ is relatively compact while $\overline{B}(\alpha(n),\overline{r}) = X$ is not compact. On the other hand, if there is one such $i$, then $B(\alpha(n_i),\overline{r_i}) = X$. Thus, in case that $X$ is not compact and $\overline{r} \geq 1$, the condition $\overline{B}(\alpha(n),\overline{r}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i),\overline{r_i})$ is fulfilled, if and only if there is some $i \in \{0,...,k\}$ with $\overline{r_i} > 1$. Hence, for the following we can assume without loss of generality that $X$ is compact or $\overline{r} < 1$. In case that $\overline{r} < 1$ we can find some $i \in \mathbb{N}$ such that $\alpha(n) \in K_i^{\circ}$ and thus $\overline{B}(\alpha(n),\overline{r}) \subseteq K_{i+1}$ by Proposition 4.4.43 of [Bra99a]. In case that $X$ is compact, there is some $i \in \mathbb{N}$ such that $X = K_{i+1}$ and thus $\overline{B}(\alpha(n),\overline{r}) \subseteq K_{i+1}$ holds in both cases (in the second case $i$ is even independent of $n$, but we will not use this property). If $\overline{r} < 1$, then let $j_0 \in \mathbb{N}$ be such that $\overline{r} + 2^{-j_0+1} < 1$ and otherwise let $j_0 := 0$. Given $i$, we can effectively find some finite subset $Q_j \subseteq \text{range}(\alpha)$ for any $j \in \mathbb{N}$ such that $d_{\mathcal{K}}(K_{i+1}, Q_j) < 2^{-j}$. In the next step, we can effectively find some subset $R_j \subseteq Q_j$ such that

$$Q_{j+1} \cap \overline{B}(\alpha(n), \overline{r} + 2^{-j}) \subseteq R_j \subseteq B(\alpha(n), \overline{r} + 2^{-j+1}) \cap Q_{j+1}.$$

Now let $L_j := \overline{\bigcup_{\iota=j}^{\infty} R_\iota}$ for any $j \in \mathbb{N}$. Then we obtain

$$\overline{B}(\alpha(n), \overline{r}) \subseteq L_j \subseteq \overline{B}(\alpha(n), \overline{r} + 2^{-j+1}) \subseteq K_{i+1}$$

for any $j \geq j_0$ and thus $L_j$ is compact. Moreover, for any $l \in \mathbb{N}$ and $j \geq j_0$ it follows

$$d_{\mathcal{K}}\left(L_j, \bigcup_{\iota=j}^{j+l+2} R_\iota\right) = \sup_{m > l+2} d'_{\mathcal{K}}\left(R_{j+m}, \bigcup_{\iota=j}^{j+l+2} R_\iota\right) \leq 2^{-j-l-1}.$$

Here, the last inequality holds, since $x \in R_{j+m}$ and $m > l+2$ implies $d(\alpha(n), x) \leq \overline{r} + 2^{-j-m+1}$. Since $x \in K_{i+1}$, there is some $y \in Q_{j+l+1}$ such that $d(x, y) \leq 2^{-j-l-1}$. Then

$$d(\alpha(n), y) \leq \overline{r} + 2^{-j-m+1} + 2^{-j-l-1} \leq \overline{r} + 2^{-j-l}.$$

Hence $y \in R_{j+l}$. This proves the inequality. Altogether, this shows that the partial function $f :\subseteq \mathbb{N} \to \mathcal{K}(X), \langle n, r, j \rangle \mapsto L_j$ is $(\delta_{\mathbb{N}}, \delta_{\mathcal{K}(X)})$–computable (in case that $X$ is compact or restricted to inputs with $\overline{r} < 1$ and $j \geq j_0$). Now we claim that

$$\overline{B}(\alpha(n), \overline{r}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i), \overline{r_i}) \iff (\exists j \geq j_0)\; L_j \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i), \overline{r_i}).$$

Since $\delta_{\mathcal{K}(X)} \leq \delta_{\text{cover}}$ (with $\delta_{\text{cover}}$ as defined in [BPar]) it follows that we can effectively enumerate all rational open covers of $\overline{B}(\alpha(n), \overline{r})$ and this implies that the set given in (2) is r.e. It remains to prove the claim. It is clear that "$\Leftarrow$" holds since $\overline{B}(\alpha(n), \overline{r}) \subseteq L_j$ for all $j \geq j_0$. In order to prove the other direction "$\Rightarrow$", let $\overline{B}(\alpha(n), \overline{r}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i), \overline{r_i})$. Since $L_j \subseteq \overline{B}(\alpha(n), \overline{r} + 2^{-j+1})$ it suffices that for some $j \geq j_0$ we can obtain $\overline{B}(\alpha(n), \overline{r} + 2^{-j+1}) \subseteq \bigcup_{i=0}^{k} B(\alpha(n_i), \overline{r_i})$. Let us assume that for any $j \geq j_0$ there is some point $x_j \in \overline{B}(\alpha(n), \overline{r} + 2^{-j+1})$ such that $x_j \in A := X \setminus \bigcup_{i=0}^{k} B(\alpha(n_i), \overline{r_i})$. Since $\overline{B}(\alpha(n), \overline{r} + 2^{-j_0+1})$ is compact and $A$ is closed, it follows that $(x_j)_{j \geq j_0}$ has a convergent subsequence with limit $x \in A$. On the other hand, $d(\alpha(n), x) \leq \overline{r} + 2^{-j+1}$ for all $j \geq j_0$ and hence $x \in \overline{B}(\alpha(n), \overline{r}) \subseteq X \setminus A$. Contradiction!

□

The proofs of (1) and (*) especially show that for recursively locally compact but non-compact spaces $\overline{B(x,1)} \subsetneq \overline{B}(x, 1)$ holds for all $x \in X$. Now we are prepared to prove our final result on the characterization of the Dedekind representations of the hyperspace $\mathcal{A}(X)$ of non-empty closed subsets[3]. We use the following definitions from [BPar, BW99].

**Definition 5.1.6 (Hyperspace representations)** Let $(X, d, \alpha)$ be a recursive metric space. We define the following representations of $\mathcal{A}(X)$:

(1) $\delta^{<}(p) = A :\Longleftrightarrow \Big(\langle n, r\rangle + 1 \in \text{range}(p) \iff A \cap B(\alpha(n), \overline{r}) \neq \emptyset\Big)$,

(2) $\delta^{>}(p) = A :\Longleftrightarrow \Big(\langle n, r\rangle + 1 \in \text{range}(p) \iff A \cap \overline{B}(\alpha(n), \overline{r}) = \emptyset\Big)$,

(3) $\delta^{=} := \delta^{<} \sqcap \delta^{>}$,

(4) $\delta^{<}_{\text{dist}}(p) = A :\Longleftrightarrow [\delta_X \to \delta_{\mathbb{R}_>}](p) = d_A : X \to \mathbb{R}_>$,

(5) $\delta^{>}_{\text{dist}}(p) = A :\Longleftrightarrow [\delta_X \to \delta_{\mathbb{R}_<}](p) = d_A : X \to \mathbb{R}_<$,

(6) $\delta^{=}_{\text{dist}}(p) = A :\Longleftrightarrow [\delta_X \to \delta_{\mathbb{R}}](p) = d_A : X \to \mathbb{R}$,

(7) $\delta_{\text{range}}(p) = A :\Longleftrightarrow \delta_X^{\infty}(p)$ is dense in $A$,

(8) $\delta_{\text{union}}\langle p, q\rangle = A :\Longleftrightarrow X \setminus A = \bigcup_{i=0}^{\infty} B(\alpha p(i), \overline{q(i)})$,

for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{A}(X)$.

Figure 5.1 (which is from [BPar]) summarizes the relations between these representations[4]. Any arrow indicates a corresponding reducibility. For recursively locally compact spaces the three horizontal layers of the diagram collapse (i.e. all vertical arrows can be inverted).

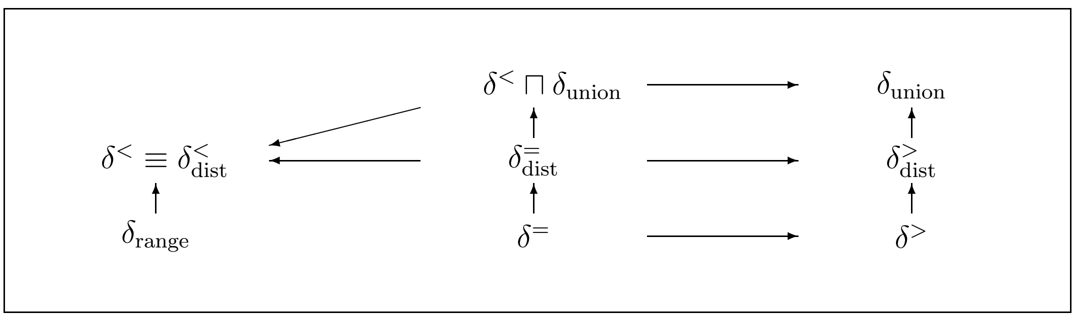


Figure 5.1: Representations of the hyperspace of closed subsets

[3]Again we consider just representations of the hyperspace of *non-empty* closed subsets, in contrast to [BPar].

[4]Figure B.1 in Appendix B displays the corresponding notions of effectivity for subsets.

**Proposition 5.1.7** *If $X$ is a nice recursively locally compact recursive metric space, then*

*(1)* $\delta^{<} \equiv \delta^{<}_{\mathrm{dist}} \equiv \delta_{\mathrm{range}}$,

*(2)* $\delta^{>} \equiv \delta^{>}_{\mathrm{dist}} \equiv \delta_{\mathrm{union}}$,

*(3)* $\delta^{=} \equiv \delta^{=}_{\mathrm{dist}} \equiv \delta^{<} \sqcap \delta_{\mathrm{union}}$.

Here, (1) follows from Theorems 3.7 and 3.8 of [BPar] since any recursively locally compact recursive metric space is complete and (2) follows from Theorems 3.9, 3.10 and 3.11 of [BPar] with the help of Lemma 5.1.5. Finally, (3) can be concluded from (1) and (2).

**Theorem 5.1.8 (Hyperspace representations)** *If $X$ is a nice recursively locally compact recursive metric space, then*

*(1)* $\delta_{\mathcal{A}(X)} \equiv \delta^{=}$,

*(2)* $\delta_{\mathcal{A}_{>}(X)} \equiv \delta^{>}$,

*(3)* $\delta_{\mathcal{A}_{<}(X)} \equiv \delta^{<}$.

**Proof.** Let $X$ be a nice recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

(1) We apply the Stability Theorem 4.5.6 in combination with the Metric Structure Theorem 4.6.7. Thus, we have to prove

(a) $d_{\mathcal{A}} : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}$ is $([\delta^{=}, \delta^{=}], \delta_{\mathbb{R}})$–computable,
(b) $\mathrm{Lim} :\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}(X)$ is $(\delta^{=\infty}, \delta^{=})$–computable,
(c) $\alpha_{\mathcal{A}} : \mathbb{N} \to \mathcal{A}(X)$ is $(\delta_{\mathbb{N}}, \delta^{=})$–computable,
(d) $\mathrm{id} : \mathcal{A}(X) \to \mathcal{A}(X)$ is $(\delta^{=}, \delta^{=})$–computable,

in order to conclude $\delta_{\mathcal{A}(X)} \equiv \delta^{=}$. Here, (d) holds obviously and (c) follows since $\alpha_{\mathcal{A}}$ is obviously $(\delta_{\mathbb{N}}, \delta_{\mathcal{K}(X)})$–computable and $\delta_{\mathcal{K}(X)} \equiv \delta_{\mathrm{Hausdorff}} \le \delta^{=}_{\mathcal{K}} \le \delta^{=}$ by Theorem 4.12 and Corollary 4.11 of [BPar] and with appropriately defined representations $\delta_{\mathrm{Hausdorff}}$ and $\delta^{=}_{\mathcal{K}}$. It remains to prove (a) and (b). Therefore, we use the fact that by Proposition 5.1.7 $\delta^{=} \equiv \delta^{=}_{\mathrm{dist}}$. Thus, it suffices to prove (a) and (b) with respect to $\delta^{=}_{\mathrm{dist}}$. Since the function $\mathcal{A}(X) \hookrightarrow \mathcal{C}(X), A \mapsto d_A$ is obviously $(\delta^{=}_{\mathrm{dist}}, [\delta_X \to \delta_{\mathbb{R}}])$–computable and hence $(\delta^{=}_{\mathrm{dist}}, \delta_{\mathcal{C}(X)})$–computable by Theorem 5.1.2 and since an analogous property holds for its partial inverse, it suffices to prove that

$$f :\subseteq \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}, (f, g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} |g - f|_{K_i},$$

defined for all functions $f, g \in \mathcal{C}(X)$ with $|f|, |g| \leq 1$, is $([\delta_{\mathcal{C}(X)}, \delta_{\mathcal{C}(X)}], \delta_{\mathbb{R}})$–computable. But this is easy to see (cf. Proposition 4.4.53 in [Bra99a]). In order to prove (b), we recall that $\frac{1}{2}d_{\mathcal{A}}(A,B) \leq d_{\mathcal{C}}(d_A, d_B) \leq d_{\mathcal{A}}(A,B)$ holds for all $A, B \in \mathcal{A}(X)$ (see Equation (4.1) in [Bra99a]). Thus, if a given sequence $(A_n)_{n\in\mathbb{N}}$ converges with a certain speed with respect to $d_{\mathcal{A}}$, then $(d_{A_n})_{n\in\mathbb{N}}$ converges at least with the same speed with respect to $d_{\mathcal{C}}$ and it suffices to compute the limit $\lim_{n\to\infty} d_{A_n} = d_{\lim_{n\to\infty} A_n}$ in $\mathcal{C}(X)$ in order to obtain the limit $\lim_{n\to\infty} A_n$ in $\mathcal{A}(X)$. But the limit in $\mathcal{C}(X)$ is obviously computable with respect to $\delta_{\mathcal{C}(X)}$ and hence the limit in $\mathcal{A}(X)$ is computable with respect to $\delta^{=}_{\text{dist}}$.

(2) We apply Proposition 3.2.4. Thus, we have to prove

(a) $d'_{\mathcal{A}} : \mathcal{A}_>(X) \times \mathcal{A}(X) \to \mathbb{R}_>, (A,B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1}|d_B \dot{-} d_A|_{K_i}$ is $([\delta^>, \delta_{\mathcal{A}(X)}], \delta_{\mathbb{R}_>})$–computable,

(b) $\text{Inf} :\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}_>(X), (B_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} B_n$ is $(\delta^{\infty}_{\mathcal{A}(X)}, \delta^>)$–computable,

in order to conclude $\delta_{\mathcal{A}_>(X)} \equiv \delta^>$. Here (b) follows easily since Inf is obviously $(\delta^{\infty}_{\text{union}}, \delta_{\text{union}})$–computable and $\delta_{\mathcal{A}(X)} \equiv \delta^{=} \leq \delta^> \equiv \delta_{\text{union}}$ by (1) and Proposition 5.1.7. It remains to prove (a). We will use that $\delta^> \equiv \delta^>_{\text{dist}}$ by Proposition 5.1.7, $\delta_{\mathcal{A}(X)} \equiv \delta^{=} \equiv \delta^{=}_{\text{dist}}$ by (1) and Proposition 5.1.7 and $\delta_{\mathcal{C}(X)} \equiv [\delta_X \to \delta_{\mathbb{R}}]$ by Theorem 5.1.2. Since $(-g) \dot{-} (-f) = f \dot{-} g$ and $f \mapsto -f$ is $([\delta_X \to \delta_{\mathbb{R}_<}], [\delta_X \to \delta_{\mathbb{R}_>}])$– and $([\delta_X \to \delta_{\mathbb{R}}], [\delta_X \to \delta_{\mathbb{R}}])$–computable (which can be proved by Proposition 5.1.1), we can conclude (a) from Lemma 5.1.3, Proposition 5.1.4 and the fact that the map $(x_i)_{i\in\mathbb{N}} \mapsto \sum_{i=0}^{\infty} 2^{-i-1}\frac{x_i}{1+x_i}$ is $(\delta^{\mathbb{N}}_{\mathbb{R}_>}, \delta_{\mathbb{R}_>})$–computable (for sequences of non-negative reals).

(3) We apply Proposition 3.2.4. Thus, we have to prove

(a) $\overline{d'_{\mathcal{A}}} : \mathcal{A}_<(X) \times \mathcal{A}(X) \to \mathbb{R}_>, (A,B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1}|d_A \dot{-} d_B|_{K_i}$ is $([\delta^<, \delta_{\mathcal{A}(X)}], \delta_{\mathbb{R}_>})$–computable,

(b) $\text{Sup} :\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}_<(X), (B_n)_{n\in\mathbb{N}} \mapsto \overline{\bigcup_{n=0}^{\infty} B_n}$ is $(\delta^{\infty}_{\mathcal{A}(X)}, \delta^<)$–computable,

in order to conclude $\delta_{\mathcal{A}_<(X)} \equiv \delta^<$. Here (b) follows since Sup is obviously $(\delta^{\infty}_{\text{range}}, \delta_{\text{range}})$–computable and $\delta_{\mathcal{A}(X)} \equiv \delta^{=} \leq \delta^< \equiv \delta_{\text{range}}$ by (1) and Proposition 5.1.7. It remains to prove (a). But this can be done similarly as in the proof of (2)(a).

□

This result proves that our Dedekind representations of the hyperspace $\mathcal{A}(X)$ are equivalent to naturally defined and well-known representations from computable analysis [BPar, BW99]. Moreover, this finally proves Proposition 3.9.6, which is a direct corollary of Theorem 5.1.8.

## 5.2 Some computable embeddings and injections

In this section we briefly mention that the hyper and function spaces used in this work are related to each other by canonical embeddings and injections. Some of these mappings have been used implicitly. We start with the distance maps.

**Theorem 5.2.1 (Distance maps)** *If $X$ is a nice recursively locally compact recursive metric space, then*

*(1)* $\mathrm{dist}: \mathcal{A}(X) \to \mathcal{C}(X), A \mapsto d_A$ *is* $(\delta_{\mathcal{A}(X)}, \delta_{\mathcal{C}(X)})$*–computable,*

*(2)* $\mathrm{dist}: \mathcal{A}_<(X) \to \mathcal{USC}(X), A \mapsto d_A$ *is* $(\delta_{\mathcal{A}_<(X)}, \delta_{\mathcal{USC}(X)})$*–computable,*

*(3)* $\mathrm{dist}: \mathcal{A}_>(X) \to \mathcal{LSC}(X), A \mapsto d_A$ *is* $(\delta_{\mathcal{A}_>(X)}, \delta_{\mathcal{LSC}(X)})$*–computable.*

*All three maps are injective and their partial inverses are computable in the corresponding sense.*

This is a direct corollary of Theorem 5.1.2, Theorem 5.1.8 and Proposition 5.1.7. Finally, we mention the graph, the epigraph and the hypograph maps for the special case of Euclidean space.

**Theorem 5.2.2 (Graph maps)** *Let* $X = \mathbb{R}^n$.

*(1)* $\mathrm{graph} : \mathcal{C}(X) \to \mathcal{A}(X \times \mathbb{R}), f \mapsto \mathrm{graph}(f) = \{(x, y) : f(x) = y\}$ *is* $(\delta_{\mathcal{C}(X)}, \delta_{\mathcal{A}(X\times\mathbb{R})})$*–computable,*

*(2)* $\mathrm{epi} : \mathcal{LSC}(X) \to \mathcal{A}_>(X \times \mathbb{R}), f \mapsto \mathrm{epi}(f) = \{(x, y) : f(x) \leq y\}$ *is* $(\delta_{\mathcal{LSC}(X)}, \delta_{\mathcal{A}_>(X\times\mathbb{R})})$*–computable,*

*(3)* $\mathrm{hypo} : \mathcal{USC}(X) \to \mathcal{A}_>(X \times \mathbb{R}), f \mapsto \mathrm{hypo}(f) = \{(x, y) : f(x) \geq y\}$ *is* $(\delta_{\mathcal{USC}(X)}, \delta_{\mathcal{A}_>(X\times\mathbb{R})})$*–computable.*

*All three maps are injective and their partial inverses are computable in the corresponding sense.*

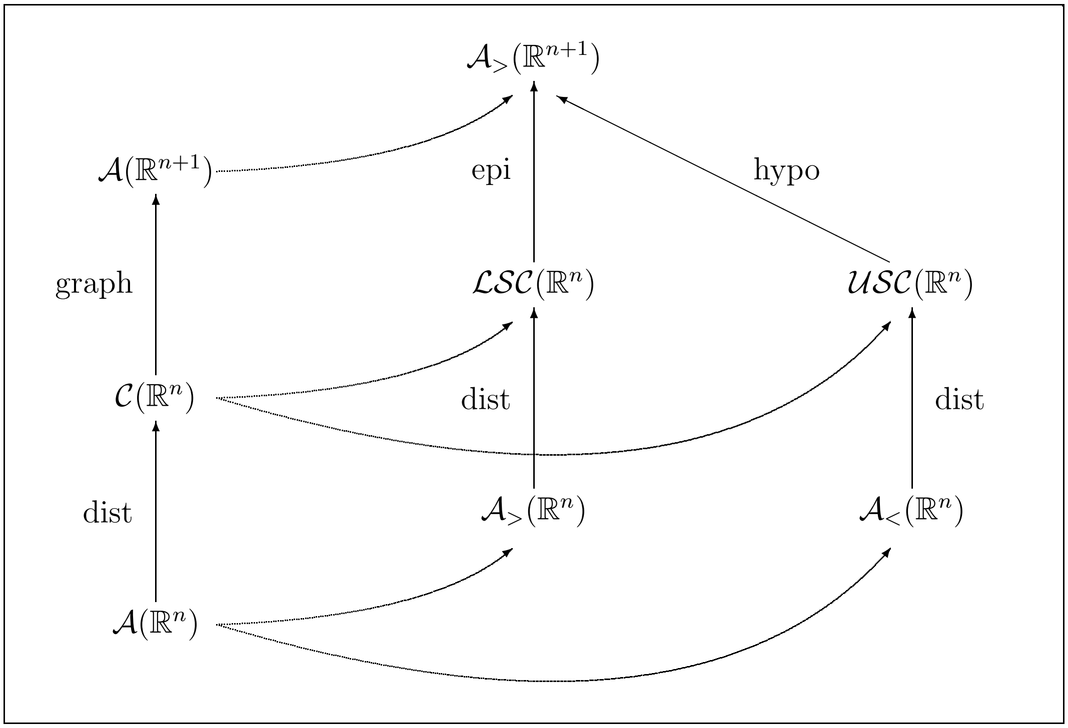


Figure 5.2: Computable embeddings and injections

For the proof of this result we can use Theorem 5.1.2, Proposition 5.1.7 and Theorem 5.1.8. Using these results, property (1) is a consequences of Theorem 8.3 in [Bra01] which shows that graph is even computable in the general case of a computable metric space $X$. However, the partial inverse of the map graph is not computable for general computable metric spaces $X$, but only under additional conditions (such as effective local connectedness of $X$, see Theorem 14.6 in [Bra01]; these additional conditions are fulfilled for $X = \mathbb{R}^n$). Properties (2) and (3) and the corresponding statement on the inverses follow from Theorem 3.8 of [WZ00] (at least for the one-dimensional case $n = 1$; we leave the general result to the reader). Together with the obvious computable injections $\mathcal{A}(X) \hookrightarrow \mathcal{A}_<(X)$, $\mathcal{A}(X) \hookrightarrow \mathcal{A}_>(X)$, $\mathcal{C}(X) \hookrightarrow \mathcal{LSC}(X)$ and $\mathcal{C}(X) \hookrightarrow \mathcal{USC}(X)$ we obtain the diagram in Figure 5.2. Any straight arrow denotes a computable embedding (i.e. a computable injection with partial computable inverse) and all other arrows denote computable injections. From Figure 5.2 one can directly conclude the interesting result that positive information on closed subsets of dimension $n$ can be embedded into negative information on closed subsets of dimension $n + 1$.

## 5.3 Effective quasi-metrizability

In this section we want to discuss the possibility of effective quasi-metrizability. A similar investigation of effective metrizability of regular spaces has been presented by Schröder [Sch98]. We first recall some basic facts on purely topological quasi-metrizability. In contrast to metrizability, which requires constructions such as Urysohn's Lemma [Eng89], it is straightforward to see that any second-countable $T_0$–space is quasi-metrizable [Smy92]. In the following we will say that $(X, B)$ is a *second-countable $T_0$–space*, if $B : \mathbb{N} \to 2^X$ is a function such that range$(B)$ is a subbase of a $T_0$–topology on $X$. For short, we write $B_i := B(i)$ and we assume that $X = \bigcup_{n=0}^{\infty} B_n$ for any such subbase. The following construction has been used for instance by Smyth [Smy92].

**Definition 5.3.1 (Associated quasi-metric)** Let $(X, B)$ be a second-countable $T_0$–space. We define subrelations $\sqsubseteq_n$ of $X \times X$ by

$$x \sqsubseteq_n y :\iff (\forall i = 0, ..., n)(y \in B_i \Longrightarrow x \in B_i)$$

for all $x, y \in X$ and $n \in \mathbb{N}$ and we define a map $d : X \times X \to \mathbb{R}$ by

$$d(x, y) := \begin{cases} \inf\{2^{-n} : n \in \mathbb{N} \text{ and } x \sqsubseteq_n y\} & \text{if the set is non-empty} \\ 2 & \text{otherwise} \end{cases}$$

for all $x, y \in X$. This map is called the *quasi-metric associated with* $(X, B)$.

In the following proposition we will show that $d$ actually is a quasi-metric. Sometimes we will use equivalence relations $\equiv_n$ which are associated with $\sqsubseteq_n$ and defined by $x \equiv_n y :\iff x \sqsubseteq_n y$ and $y \sqsubseteq_n x$ for all $x, y \in X$ and $n \in \mathbb{N}$. If $d_\star$ denotes the metric associated with $d$, then we obtain the following helpful properties:

$$d(x, y) \leq 2^{-n} \iff x \sqsubseteq_n y \quad \text{and} \quad d_\star(x, y) \leq 2^{-n} \iff x \equiv_n y$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Now we are prepared to prove the following proposition which is essentially due to Smyth [Smy92]. We include the proofs for completeness.

**Proposition 5.3.2** *Let $(X, B)$ be a second-countable $T_0$–space with associated quasi-metric $d : X \times X \to \mathbb{R}$. Then*

*(1) $d$ is a quasi-metric,*

*(2) the upper topology of $d$ on $X$ coincides with the topology induced by the subbase* range$(B)$*,*

*(3) $(X, d_\star)$ is a separable metric space.*

**Proof.**

(1) Obviously $x \sqsubseteq_n x$ for all $n \in \mathbb{N}$ and $x \in X$ and thus $d(x,x) = 0$. If $x, y \in X$ are such that $d(x,y) = d(y,x) = 0$, then $x \sqsubseteq_n y$ and $y \sqsubseteq_n x$ for all $n \in \mathbb{N}$. Thus, for any $i \in \mathbb{N}$ we obtain $x \in B_i \iff y \in B_i$. Since $B$ enumerates a subbase of a $T_0$-topology, it follows that $x = y$. Let $x, y, z \in X$ and let $n$ and $k$ be numbers such that $x \sqsubseteq_n z$ and $z \sqsubseteq_k y$, respectively. Let $m := \min\{n,k\}$. Then $x \sqsubseteq_m z$ and $z \sqsubseteq_m y$ and thus $x \sqsubseteq_m y$. Hence $d(x,y) \le \max\{d(x,z), d(z,y)\} \le d(x,z) + d(z,y)$.

(2) Let $x \in X$, $0 < \varepsilon \le 2$ and $m = \min\{n : 2^{-n} < \varepsilon\}$. Then we obtain

$$\begin{aligned}
& B_>(x,\varepsilon) \\
= \; & \{y \in X : d(y,x) < \varepsilon\} \\
= \; & \{y \in X : (\exists n)(2^{-n} < \varepsilon \text{ and } y \sqsubseteq_n x)\} \\
= \; & \{y \in X : 2^{-m} < \varepsilon \text{ and } (\forall i = 0, ..., m)(x \in B_i \Longrightarrow y \in B_i)\} \\
= \; & \begin{cases} \bigcap\{B_i : i \le m \text{ and } x \in B_i\} & \text{if } x \in \bigcup_{i=0}^m B_i \\ X & \text{otherwise} \end{cases} .
\end{aligned}$$

If $\varepsilon > 2$, then $B_>(x,\varepsilon) = X$. Hence, any open set with respect to the upper topology is an open set with respect to the topology induced by range($B$).

Now, let in turn $U$ be open with respect to the topology induced by the subbase range($B$) and let $x \in U$. Then there are $i_1, ..., i_k \in \mathbb{N}$ such that $x \in \bigcap_{j=1}^k B_{i_j} \subseteq U$. Now let $n := \max\{i_1, ..., i_k\}$ and $\varepsilon := 2^{-n+1}$. Then $x \in B_>(x,\varepsilon) = \bigcap\{B_i : i \le n \text{ and } x \in B_i\} \subseteq \bigcap_{j=1}^k B_{i_j} \subseteq U$. Thus, $U$ is also open with respect to the upper topology.

(3) For any $n \in \mathbb{N}$ the equivalence relations $\equiv_n$ induce a partioning of $X$ into finitely many classes. If we select one member of each of these classes for any $n \in \mathbb{N}$, then we obtain a countable set $S \subseteq X$. Now, for any $x \in X$ and $\varepsilon > 0$, there is some $n \in \mathbb{N}$ with $2^{-n} < \varepsilon$ and some $y \in S$ such that $x \equiv_n y$ and hence $d_\star(x,y) \le 2^{-n} < \varepsilon$. Thus, $S$ is dense in $X$ with respect to $d_\star$.

□

The proof of (1) even shows that $d$ is a non-Archimedean quasi-metric and a similar proof as the one of (3) shows that $d$ is totally bounded [Smy92]. One could ask whether the special construction of the associated quasi-metric implies some additional nice continuity properties of $d$. The following example shows that the associated quasi-metric is not necessarily continuous from above (which by Proposition 2.3.4 is equivalent to the fact that it is not necessarily consistent from above with respect to the upper topology).

**Example 5.3.3** *Let* $X := \{-1\} \cup \{\frac{1}{2^{-n}} : n \in \mathbb{N}\}$ *and let* $B_0 := \{-1\}$ *and* $B_{i+1} := \{-1\} \cup \{\frac{1}{2^{-n}} : n \geq i\}$ *for all* $i \in \mathbb{N}$*. Then* $(X, B)$ *is a second-countable* $T_0$*–space with associated quasi-metric* $d : X \times X \to \mathbb{R}$ *and we directly obtain* $x \sqsubseteq y \iff (\forall i)(y \in B_i \Longrightarrow x \in B_i) \iff x \leq y$ *for the associated partial order* $\sqsubseteq$*. Hence, by* $x_n := \frac{1}{2^{-n}}$ *a sequence* $(x_n)_{n\in\mathbb{N}}$ *with* $x := \inf_{n\in\mathbb{N}} x_n = -1$ *is defined. Thus,* $(X, X, d)$ *is not continuous from above.*

Especially, this shows that in the setting of the example $\text{Inf} :\subseteq X^{\mathbb{N}} \to X$ is not continuous. We will even use a slightly stronger continuity property for the infimum in the following which we define next.

**Definition 5.3.4 (Strong continuity)** Let $(X, d)$ be a quasi-metric space. The corresponding infimum map $\text{Inf} :\subseteq X^{\mathbb{N}} \to X$ is called *strongly continuous*, if for any sequence $(x_i)_{i\in\mathbb{N}}$ such that $x = \inf_{i\in\mathbb{N}} x_i$ exists and any open set $U \subseteq X$ with respect to the upper topology and with $x \in U$, there is some $i \in \mathbb{N}$ such that $x_i \in U$.

It is easy to see that any strongly continuous infimum map is especially continuous.

**Lemma 5.3.5** *Let* $(X, d)$ *be a quasi-metric space. If* $\text{Inf} :\subseteq X^{\mathbb{N}} \to X$ *is strongly continuous, then it is also continuous with respect to the upper topology.*

**Proof.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$ such that $x := \inf_{n\in\mathbb{N}} x_n$ exists and let $U \subseteq X$ be an open neighbourhood of $x$. Then there is some $i \in \mathbb{N}$ such that $x_i \in U$ and thus $V := X^i \times U \times X \times X \times ...$ is an open neighbourhood of $(x_n)_{n\in\mathbb{N}}$. Let $(y_n)_{n\in\mathbb{N}} \in V$ be such that $y := \inf_{n\in\mathbb{N}} y_n$ exists. Then we obtain $y \sqsubseteq y_i \in U$ and thus $y \in U$, since $d(y, y_i) = 0$ and there is some $\varepsilon > 0$ such that $B_>(y_i, \varepsilon) \subseteq U$. Altogether, $\text{Inf}(V) \subseteq U$ and hence Inf is continuous. □

In general, continuity does not imply strong continuity, not even in case of the induced quasi-metric, as the next example shows.

**Example 5.3.6** *Let* $X := \mathbb{N}$ *and let* $B_i := \{0\} \cup \{i\}$ *for all* $i \in \mathbb{N}$*. Then* $(X, B)$ *is a second-countable* $T_0$*–space with associated quasi-metric* $d : X \times X \to \mathbb{R}$ *and we directly obtain* $x \sqsubseteq y \iff (x = 0 \text{ or } x = y)$ *for the associated partial order* $\sqsubseteq$*. Thus, if* $(x_n)_{n\in\mathbb{N}}$ *is not constant, then* $\inf_{n\in\mathbb{N}} x_n = 0$*. Hence,* Inf *is not strongly continuous, but* Inf *is continuous (and* $(X, X, d)$ *is continuous from above).*

The next example shows that the induced quasi-metric space is not necessarily strongly dense in itself.

**Example 5.3.7** *Let* $X := \{A_n : n \in \mathbb{N}\}$ *with* $A_n := \{n, n+1, n+2, ..., 2n\}$ *for all* $n \in \mathbb{N}$ *and let* $B_i := \{A \in X : i \in A\}$ *for all* $i \in \mathbb{N}$. *Then* $(X, B)$ *is a second-countable* $T_0$*–space with associated quasi-metric* $d : X \times X \to \mathbb{R}$ *and we directly obtain* $d(A_i, A_j) \leq 2^{-n} \iff A_i \sqsubseteq_n A_j \iff A_j \cap \{0, ..., n\} \subseteq A_i$. *We claim that* $X$ *is not strongly dense in* $(X, d)$. *Let us assume that* $X$ *would be strongly dense with constant* $c \geq 1$ *and let* $U := A_c$, $V := A_{c+1}$ *and* $\varepsilon := \frac{1}{2^{2c}}$. *Then there exists some* $W \in X$ *such that*

$$d(U, W) < \varepsilon \text{ and } d_\star(W, V) < c \cdot d(U, V) + \varepsilon.$$

*Thus, there is some* $k \in \mathbb{N}$ *with* $W = A_k$ *and* $d(A_c, A_k) \leq \frac{1}{2^{2c+1}}$ *and consequently* $k = c$ *or* $k > 2c + 1$. *In the first case, we obtain* $W = U$ *and* $d_\star(W, V) = d_\star(U, V) \geq d(V, U) = d(A_{c+1}, A_c) = \frac{1}{2^{c-1}}$ *and in the second case* $d_\star(W, V) \geq d(W, V) = d(A_k, A_{c+1}) = \frac{1}{2^c}$. *Hence, in both cases*

$$\begin{aligned} c \cdot d(U, V) + \varepsilon &= c \cdot d(A_c, A_{c+1}) + \frac{1}{2^{2c}} \\ &\leq \frac{c}{2^{2c}} + \frac{1}{2^{2c}} \\ &= \frac{c+1}{2^{2c}} \\ &\leq \frac{1}{2^c} \\ &\leq d_\star(W, V). \end{aligned}$$

*Contradiction!*

If $(X, B)$ is a second-countable $T_0$–space, then the induced quasi-metric $d$ yields an upper generated quasi-metric space $(X, X, d)$ and $(X, d_\star, \alpha)$ is a separable metric space where $\alpha : \mathbb{N} \to X$ is some function such that range($\alpha$) is dense in $X$. Hence we obtain a Cauchy representation $\delta_X = \text{Lim} \circ \alpha^{\mathbb{N}}$ and a corresponding upper Dedekind representation $\delta_{X_>} = \text{Inf} \circ \delta_X^\infty$ of $X$. On the other hand, any second-countable $T_0$–space can be represented by its standard representation (see [Wei00]) which we will define next.

**Definition 5.3.8 (Standard representation)** Let $(X, B)$ be a second-countable $T_0$–space. The *standard representation* $\delta_B$ of $X$ is defined by

$$\delta_B(p) = x :\iff \text{range}(p) = \{n \in \mathbb{N} : x \in B_n\}$$

for all $p \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$.

Thus, $\delta_B(p) = x$, if $p$ is a list of all "properties" $B_n$ of $x$. It is known that any representation $\delta$ of a second-countable $T_0$–space $(X, B)$ is admissible, if and only if it is topologically equivalent to $\delta_B$, i.e. if $\delta \equiv_{\rm t} \delta_B$, see [Wei00]. Thus, we obtain the following corollary of Theorem 3.3.2 on admissibility of the upper Dedekind representation.

**Corollary 5.3.9** *Let $(X, X, d, \alpha)$ be a semi-recursive quasi-metric space and let $B$ be some enumeration of a subbase of the upper topology of this space. Then $\delta_{X_>} \equiv_{\rm t} \delta_B$.*

Now we want to investigate a kind of an inverse statement. What happens if we start with the subbase $B$ and consider the induced quasi-metric $d$. Under which circumstances does $\delta_{X_>} \equiv \delta_B$ hold? The following theorem gives some sufficient conditions for this equivalence.

**Theorem 5.3.10 (Effective quasi-metrizability)** *Let $(X, B)$ be a second-countable $T_0$–space and let $d$ be the associated quasi-metric. Then:*

*(1) $\delta_{X_>} \leq_{\rm t} \delta_B$, if $\mathrm{Inf} :\subseteq X^{\mathbb{N}} \to X$ is strongly continuous,*

*(2) $\delta_B \leq_{\rm t} \delta_{X_>}$, if $X$ is strongly dense in $(X, d)$ and $(X, d_\star)$ is complete.*

*In (1) and (2) the reducibility is even computable, if a function $\alpha : \mathbb{N} \to X$ with $\mathrm{range}(\alpha)$ dense in $(X, d_\star)$ is used for the definition of $\delta_{X_>}$ such that the set $A := \{(i, j) \in \mathbb{N}^2 : \alpha(i) \in B_j\}$ is r.e. and recursive, respectively.*

**Proof.** By Proposition 5.3.2 there exists a function $\alpha : \mathbb{N} \to X$ such that $\mathrm{range}(\alpha)$ is dense in $X$ with respect to the metric $d_\star$. Let $\delta_X = \mathrm{Lim} \circ \alpha^{\mathbb{N}}$ be the corresponding Cauchy representation and let $\delta_{X_>} = \mathrm{Inf} \circ \delta_X^\infty$ be the associated Dedekind representation.

(1) We describe a Turing machine $M$ with an enumeration of $A$ as oracle which translates $\delta_{X_>}$ to $\delta_B$. Let $p = \langle p_0, p_1, p_2, ...\rangle$ be such that $\delta_{X_>}(p) = x$. Then any $p_i$ is a list of numbers $n_{ij}$ such that $\lim_{j\to\infty} \alpha(n_{ij}) = \delta_X(p_i) =: x_i$ and $\inf_{i\in\mathbb{N}} x_i = x$. Moreover, $d_\star(x_i, \alpha(n_{ij})) \leq 2^{-j}$ for all $i, j \in \mathbb{N}$. Thus, $x \sqsubseteq_j x_i \equiv_j \alpha(n_{ij})$ for all $i, j \in \mathbb{N}$ and hence $\alpha(n_{ij}) \in B_k$ implies $x \in B_k$ for all $k = 0, ..., j$.

Now machine $M$ with an enumeration of $A$ as oracle and input $p$ operates in parallel phases $\langle i, j, k\rangle = 0, 1, 2, ...$ as follows: if $k \leq j$ and $\alpha(n_{ij}) \in B_k$ is enumerated by the oracle, then the machine writes $k$ on its output tape.

It is clear that in this way $x \in B_k$ for any $k$ which has been written on the output tape by $M$. We still have to prove that $M$ also writes $k$ on its output tape for any $k$ such that $x \in B_k$. But since $x = \inf_{i\in\mathbb{N}} x_i$ and Inf is strongly continuous, it follows that there is some $i \in \mathbb{N}$ with $x_i \in B_k$ and thus there is some $j \geq k$ such that $\alpha(n_{ij}) \in B_k$. Hence, in phase $\langle i, j, k\rangle$ machine $M$ will finally write $k$ on its output tape.

Altogether, this shows that $M$ with an enumeration of $A$ as oracle computes a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\delta_{X_>}(p) = \delta_B F(p)$ for all $p \in \mathrm{dom}(\delta_{X_>})$. In any case $F$ is continuous and if $A$ is r.e., then $F$ is even computable.

(2) Now let $X$ be strongly dense in $(X, d)$ with constant $c$ and let $(X, d_\star)$ be complete. We describe a Turing machine $M$ with oracle $A$ which translates $\delta_B$ to $\delta_{X_>}$. Let $p \in \mathrm{dom}(\delta_B)$ such that $\delta_B(p) = x$. Then $p$ is a list of all numbers $j$ such that $x \in B_j$. Since

$$\begin{aligned} & d(x, \alpha(m)) < 2^{-n+1} \\ \iff\ & d(x, \alpha(m)) \leq 2^{-n} \\ \iff\ & x \sqsubseteq_n \alpha(m) \\ \iff\ & (\forall i = 0, ..., n)(\alpha(m) \notin B_i \text{ or } x \in B_i) \end{aligned}$$

for all $m, n \in \mathbb{N}$ and analogously

$$\begin{aligned} & d_\star(\alpha(k), \alpha(m)) < 2^{-n+1} \\ \iff\ & d_\star(\alpha(k), \alpha(m)) \leq 2^{-n} \\ \iff\ & \alpha(k) \equiv_n \alpha(m) \\ \iff\ & (\forall i = 0, ..., n)(\alpha(m) \in B_i \iff \alpha(k) \in B_i) \end{aligned}$$

it follows that the set $\{(m, n) \in \mathbb{N}^2 : d(x, \alpha(m)) < 2^{-n+1}\}$ and the set $\{(k, m, n) \in \mathbb{N}^3 : d_\star(\alpha(k), \alpha(m)) < 2^{-n+1}\}$ are r.e. relatively to input $p$ and oracle $A$.

Thus, the Turing machine $M$ with oracle $A$ and input $p$ can find all values $l \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that $d(x, \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$. Given one such pair, the machine can effectively find a sequence $(m_k)_{k\in\mathbb{N}}$ such that

$$d_\star(\alpha(m_{k+1}), \alpha(m_k)) < 2^{-k-l-1} \text{ and } d(x, \alpha(m_k)) < \frac{1}{c} \cdot 2^{-k-l-2} \quad (*)$$

for all $k \in \mathbb{N}$. By induction one can prove that such a sequence exists since $X$ is strongly dense with constant $c$ and $\alpha$ is dense in $(X, d_\star)$. For any starting pair $(l, m_0)$ the machine $M$ compiles some sequence $q_i = m_0 m_1 m_2 ...$ and writes $q = \langle q_0, q_1, q_2, ...\rangle$ on the output tape.

We still have to show that $\delta_{X_>}(q) = x$. First of all we prove $q_i \in \mathrm{dom}(\delta_X)$ and $x \sqsubseteq \delta_X(q_i)$ for any $i \in \mathbb{N}$. This can be done analogously as in the proof of Proposition 3.6.1: with the notations as above $(\alpha(m_k))_{k\in\mathbb{N}}$ is a Cauchy sequence and since $(X, d_\star)$ is complete, the limit $y := \lim_{k\to\infty} \alpha(m_k) = \delta_X(q_i)$ exists in $X$. We obtain

$$d(x, y) \leq d(x, \alpha(m_k)) + d_\star(\alpha(m_k), y) < \frac{1}{c} \cdot 2^{-k-l-2} + 2^{-k-l} < 2^{-k-l+1}$$

for all $k \in \mathbb{N}$ and hence $d(x, y) = 0$ and $x \sqsubseteq y$. For any $l$ and any starting point $m_0$ with $d(x, \alpha(m_0)) < \frac{1}{c} \cdot 2^{-l-2}$ we have found a point $y$. Let $V$ be the set of all these points, i.e. $V = \{\delta_X(q_i) : i \in \mathbb{N}\}$. As in the proof of Proposition 3.6.1 one can show that $V$ is dense in $\{y \in X : x \sqsubseteq y\}$ and by Lemma 2.4.2 this implies $\delta_B(p) = x = \inf V = \mathrm{Inf} \circ \delta_X^\infty(q) = \delta_{X_>}(q)$.

Altogether, this shows that machine $M$ with oracle $A$ computes a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\delta_B(p) = \delta_{X_>} F(p)$ for all $p \in \mathrm{dom}(\delta_B)$. In any case $F$ is continuous and if $A$ is recursive, then $F$ is even computable.

□

We should mention that the purely topological part of the statement of the previous theorem does obviously not depend on the choice of $\alpha$. Given a second-countable $T_0$–space which fulfills the requirements given in the theorem, how far is the induced space $(X, X, d, \alpha)$ from being a (strongly) semi-recursive quasi-metric space? The following corollary gives an answer to this question.

**Corollary 5.3.11** *Let $(X, B)$ be a second-countable $T_0$–space, let $d$ be the associated quasi-metric and let $\alpha : \mathbb{N} \to X$ be a function such that* $\mathrm{range}(\alpha)$ *is dense in $(X, d_\star)$. If*

*(1) $X$ is strongly dense in $(X, d)$,*

*(2) $\mathrm{Inf} :\subseteq X^\mathbb{N} \to X$ is strongly continuous,*

*(3) $(X, d_\star)$ is complete,*

*(4) $A := \{(i, j) \in \mathbb{N}^2 : \alpha(i) \in B_j\}$ is recursive,*

*then $(X, X, d, \alpha)$ is a strong semi-recursive quasi-metric space and $\delta_B \equiv \delta_{X_>}$ holds for the upper Dedekind representation $\delta_{X_>}$ of this space.*

For the proof one has to show that $d : X \times X \to \mathbb{R}_>$ is $([\delta_{X_>}, \delta_X], \delta_{\mathbb{R}_>})$–computable, and $d_\star : X \times X \to \mathbb{R}_>$ is $([\delta_X, \delta_X], \delta_{\mathbb{R}_>})$–computable. But this has implicitly been done in the proof of the previous theorem. Alternatively, one could also derive the equivalence statement of the previous corollary from Proposition 3.2.4 which characterizes the computational equivalence class of the upper Dedekind representation. However, the previous theorem contains some additional information. The following example shows that there are strong recursive quasi-metric spaces such that Inf is not strongly continuous.

**Example 5.3.12** *We consider the metric space* $\mathbb{R}$ *and the strong recursive quasi-metric space* $(\mathcal{K}(\mathbb{R}), \mathcal{K}(\mathbb{R}), d'_{\mathcal{K}})$. *Let* $A_n := \{2^{-n}\} \cup \{2\}$ *and* $A := [0, 1]$. *Then* $\bigcap_{n=0}^{\infty} A_n = \{2\}$ *and hence* $A \cap \bigcap_{n=0}^{\infty} A_n = \emptyset$, *but* $A \cap A_n = \{2^{-n}\} \neq \emptyset$ *for all* $n \in \mathbb{N}$. *Thus,* Inf *is not strongly continuous.*

This example leads to the question how the continuity condition on Inf could be weakened? Many other questions concerning the effective quasi-metrization could be discussed. However, we will close our investigation at this point and leave these questions open for future research.

# Chapter 6

# Conclusion

In this work we have developed a theory of computability on quasi-metric spaces. We have seen that many spaces which have been used in computable analysis can be induced by quasi-metrics in a very natural way. Finally, we have presented a theory of quasi-metric data structures. This theory can be considered as a foundation of a more general theory of continuous data structures which implicitly is based on a number of theses which can be made explicit at this point.

First of all, it is a fundamental observation that computability implies continuity. This observation has not only be made in computable analysis [Wei00] but in many other fields such as domain theory (see [Sco70, Smy92]). Following this observation, we can formulate the first thesis.

**Thesis 1.** Any computation over topological data structures is continuous, if described in the appropriate way.

The basic intuition behind this thesis is that computers are devices which are only capable to handle finite information and thus any finite portion of information included in a computed result can only depend on a finite portion of information on the input. But this is nothing but continuity!

The next point is that we want to cover *extensional* as well as *non-extensional* computations on topological data structures. This is because it is known that there are many problems, such as determination of zeros, which are solvable only by non-extensional computations. The results of such computations do not only depend on the input but on its internal representation. If we want to hide the internal representation, then one way to achieve this is to use multi-valued operations. The corresponding notion of continuity is the notion of *lower semi-continuity* since it reflects the fact that we want to obtain positive information on our results (in contrast to that, upper semi-continuity would correspond to negative information). Hence, we arrive at our second thesis.

**Thesis 2.** Extensional and non-extensional computations can be described by lower semi-continuous single- and multi-valued operations, respectively.

The main insight which has been provided by this work is that data structures for many topological spaces can be very easily derived from corresponding quasi-metrics. A topological space without any additional structure does not offer any obvious method to single out natural operations on the space which could be used to build data structures. In the quasi-metric case the quasi-metric itself and the induced infimum operation are building blocks of suitable data structures (similarly, as the metric and the corresponding limit operation are building blocks in the metric case). For many concrete spaces these operations turn out to be very natural (such as union or intersection on hyperspaces). Altogether, we have justified our third thesis.

**Thesis 3.** Metric and quasi-metric spaces provide canonical operations to construct sound and complete topological data structures for symmetric and asymmetric spaces, respectively.

Here, *soundness* and *completeness* means that the corresponding structures reflect the right computational power with respect to more concrete models (such as Turing machines with representations). And a precise definition of soundness and completeness requires, of course, a definition of the abstract programming language which is used together with the topological data structures. With the recursive closure schemes we have presented a very rudimentary programming language which rather corresponds to functional languages than to imperative programming languages. It is obvious that other languages, such as WHILE programming languages, do much more appropriately reflect the practice of numerical programming. And actually, such languages can and have been used in similar contexts [TZ99]. However, we claim that our recursive closure schemes are the set-theoretical building blocks of such more practical programming languages. If, for instance, one would like to define the concatenation of commands in a WHILE programming language precisely, then one has to clarify the definition of composition. Finally, we arrive at our fourth thesis.

**Thesis 4.** The recursive closure schemes reflect the set-theoretical essence of realistic abstract programming languages over topological data structures.

Thus, we do not suggest these closure schemes for practical programming purposes, we just claim that they are an appropriate mathematical tool to study the set-theoretical content of such programming languages (without all those technical complications which could be induced by real world programming languages).

Of course, our approach towards a theory of continuous data structures is not the only possible approach. Several other approaches exist and we just mention the PCF approach of Martín Escardó [Esc96] and the WHILE programming language of Tucker and Zucker [TZ99]. All these approaches can be distinguished by the corresponding degree of abstractness. More concrete models, like Turing machines, operate on bits (of sequences which represent real numbers, for instance) and more abstract models consider real numbers as entities. The domain theoretic approach offers a kind of a "semi-abstract" step in between: the real numbers as entities are replaced by the domain of real numbers (which additionally includes finite approximations, i.e. rational intervals, as entities). See [SHT99, TZ01, BES02] for discussions of concrete versus abstract models. In Table 6.1 we summarize some terms which are related to the concrete and the abstract world.

| description | objects | view | effectivity |
|---|---|---|---|
| abstract | entities | extensional | recursive |
| concrete | represented | intensional | computable |

Table 6.1: The abstract and concrete world

Of course, the theory presented in this work is by no means comprehensive and there remain many open problems for future research. We give a survey of some of them.

- **Completeness and redundancy.** Our abstract language of recursive closure schemes comprises ten schemes which reflect certain elementary actions on computable operations. However, we have not proved that these schemes are non-redundant. Although unlikely, it could be possible that some of the schemes could be eliminated without any loss of generality. Especially, the sequentialization scheme has only been applied for quasi-metric structures (and has not been required for metric structures). Is it really necessary in the quasi-metric case? And even more interesting: can it happen that one has to extend the closure schemes even further if one wants to extend the range of applications to other more general structures? Is there a general completeness or incompleteness theorem in this direction?

- **Non-separable spaces.** Our definition of a perfect topological structure implies that the underlying topological spaces have to be second countable (see remark after Theorem 4.6.3). Matthias Schröder recently extended the theory of admissible representations to larger classes of spaces [Sch01, Sch02]. One important feature of such an extension is that it leads to Cartesian closed categories of admissibly represented spaces. Is it possible to extend the theory of perfect topological structures to such spaces as well?

- **Complexity.** It is still a challenging open problem to extend the definition of (polynomial time) complexity to large classes of topological spaces. While there is a very suitable definition in case of Euclidean space [Ko91, Wei00], there is no such notion in the general metric case. It seems to be possible to extend complexity theory to compact or even locally compact separable metric spaces [Sch95, Wei02], but a more general approach is out of sight. There is even no general and uniform definition of complexity for computable functions of type $f : \mathcal{C}[0,1] \to \mathbb{R}$ (since $\mathcal{C}[0,1]$ is not locally compact). Since even the symmetric case cannot be handled in full generality, it seems to be far more problematic to extend the notion of complexity to the asymmetric case (this corresponds to the fact that classically complexity is introduced for recursive sets and not for r.e. sets).

It should be mentioned that many other obvious questions have not been discussed in this work. For instance, quasi-metric spaces are closely related to the theory of cpo's [AJ94, SHLG94] (actually, they induce a kind of a quantitative partial order) and hence they are often considered as common generalization of the theories of metric and partial order based denotational semantics [Smy87, Sün95, KS02]. We have not touched these questions at all since our main goal was to exploit quasi-metric spaces for the purposes of computable analysis.

# Appendix A

# Sequential Iteration

The aim of this appendix is to introduce a sequential iteration closure scheme for computable multi-valued operations. We start with the definition.

**Definition A.1 (Sequential iteration)** If $f :\subseteq X \rightrightarrows X$ is an operation, then the *sequential iteration* $f^\nabla :\subseteq X \rightrightarrows X^\mathbb{N}$ is defined by

$$f^\nabla(x) := \{(x_n)_{n\in\mathbb{N}} : x_0 = x \text{ and } (\forall n)\ x_{n+1} \in f(x_n)\}$$

for all $x \in \mathrm{dom}(f^\nabla) := \{x \in X : (\forall n)\ x \in \mathrm{dom}(f^n)\}$.

The ordinary iteration closure scheme uses all possible results on a certain step of iteration to determine the values for the next step. In contrast to that, the sequential iteration scheme collects only "consistent sequences" such that any value in the sequence is definitely a successor of the previous value. This can make a difference as soon as one considers operations $f$ which are not single-valued (see the discussion below). We start with the proof that sequential iteration preserves computability.

**Proposition A.2 (Closure under sequential iteration)** *Let $(X, \delta_X)$ be a represented space. If $f :\subseteq X \rightrightarrows X$ is $(\delta_X, \delta_X)$–computable, then the operation $f^\nabla :\subseteq X \rightrightarrows X^\mathbb{N}$ is $(\delta_X, \delta_X^\infty)$–computable. A corresponding property holds for strong computability.*

**Proof.** Let $f :\subseteq X \rightrightarrows X$ be (strongly) $(\delta_X, \delta_X)$–computable via a computable function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. Define $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ inductively by $G\langle p, \langle q_0, q_1, ...\rangle\rangle := \langle r_0, r_1, r_2, ...\rangle$ and

$$\begin{cases} r_0 & := & p \\ r_{n+1} & := & F\langle r_n, q_n\rangle \end{cases}$$

for all $p, q_0, q_1, ... \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$. Then $G$ is computable and we claim that $f^\nabla :\subseteq X \rightrightarrows X^\mathbb{N}$ is (strongly) $(\delta_X, \delta_X^\infty)$–computable via $G$. For the proof let

$p \in \mathrm{dom}(f^\nabla \delta_X)$ and $x := \delta_X(p)$. Then $x \in \mathrm{dom}(f^\nabla)$ and $\langle p, \mathbb{N}^\mathbb{N}\rangle \subseteq \mathrm{dom}(\delta_X F)$. Now let $q_n \in \mathbb{N}^\mathbb{N}$, $n \in \mathbb{N}$. Let $r_n \in \mathbb{N}^\mathbb{N}$ be defined as above and let $x_0 := x$, $x_n := \delta_X(r_n)$ for all $n \in \mathbb{N}$. By induction one can prove $x_{n+1} \in f(x_n) = \{\delta_X F\langle r_n, q\rangle : q \in \mathbb{N}^\mathbb{N}\}$ for all $n \in \mathbb{N}$. Hence $\langle p, \mathbb{N}^\mathbb{N}\rangle \subseteq \mathrm{dom}(\delta_X^\infty G)$ and

$$
\begin{aligned}
& \{\delta_X^\infty G\langle p, q\rangle : q \in \mathbb{N}^\mathbb{N}\} \\
= \;& \{\delta_X^\infty G\langle p, \langle q_0, q_1, ...\rangle\rangle : q_0, q_1, ... \in \mathbb{N}^\mathbb{N}\} \\
= \;& \{(\delta_X(p), \delta_X F\langle r_0, q_0\rangle, \delta_X F\langle r_1, q_1\rangle, ...) : q_0, q_1, ... \in \mathbb{N}^\mathbb{N}\} \\
= \;& \{(x_n)_{n\in\mathbb{N}} : x_0 = \delta_X(p) \text{ and } (\exists q_0, q_1, ... \in \mathbb{N}^\mathbb{N})(\forall n)\; x_{n+1} = \delta_X F\langle r_n, q_n\rangle\} \\
= \;& \{(x_n)_{n\in\mathbb{N}} : x_0 = \delta_X(p) \text{ and } (\forall n)\; x_{n+1} \in f(x_n)\} \\
= \;& f^\nabla \delta_X(p).
\end{aligned}
$$

Thus, $f^\nabla$ is $(\delta_X, \delta_X^\infty)$–computable via $G$.

Now, let us consider the case of strong computability and let $p \in \mathrm{dom}(\delta_X) \setminus \mathrm{dom}(f^\nabla \delta_X)$. Then there is some $(x_n)_{n\in\mathbb{N}} \in X^\mathbb{N}$ such that $x_0 = x = \delta_X(p)$ and some $i \in \mathbb{N}$ such that $(\forall n < i)\; x_{n+1} \in f(x_n)$ and $x_i \not\in \mathrm{dom}(f)$. Thus, there are $r_0, ..., r_i \in \mathbb{N}^\mathbb{N}$ and $q_0, ..., q_i \in \mathbb{N}^\mathbb{N}$ such that $\delta_X(r_n) = x_n$ for all $n \leq i$ and $r_{n+1} = F\langle r_n, q_n\rangle$ for all $n < i$. Since $x_i \not\in \mathrm{dom}(f)$, we can conclude $\langle r_i, \mathbb{N}^\mathbb{N}\rangle \not\subseteq \mathrm{dom}(F)$ and thus $\langle p, \mathbb{N}^\mathbb{N}\rangle \not\subseteq \mathrm{dom}(G)$. Thus, $f^\nabla$ is strongly $(\delta_X, \delta_X^\infty)$–computable via $G$. □

Now the question arises, how sequential iteration is related to other closure schemes. It is easy to see that the ordinary iteration scheme can be simulated in presence of sequential iteration, since for any operation $f :\subseteq X \rightrightarrows X$ one obtains $(f^\nabla)_*(x, n) = f^*(x, n)$ for all $x \in \mathrm{dom}(f^\nabla)$ and $n \in \mathbb{N}$. On the other hand, for single-valued functions $f :\subseteq X \to X$ we obtain $[f^*] = f^\nabla$. Thus, in this case sequential iteration can also be obtained from ordinary iteration. In the case of multi-valued operations $f :\subseteq X \rightrightarrows X$ only $f^\nabla(x) \subseteq [f^*](x)$ holds for all $x \in \mathrm{dom}(f^\nabla)$ but "$\supseteq$" does not hold in general. However, for certain multi-valued operations sequential iteration can also be constructed with the help of ordinary iteration and sequentialization. This is the case at least for such operations which are multi-valued only in a natural number component (which is made precise by the following proposition). We recall that

$$\langle n, k\rangle := \frac{1}{2}(n + k)(n + k + 1) + k$$

and we define inductively $\langle n_1, ..., n_{i+1}\rangle := \langle\langle n_1, ..., n_i\rangle, n_{i+1}\rangle$ for all $n, k \in \mathbb{N}$ and $i \geq 1$.

**Proposition A.3** *Let $S$ be a natural structure. If $f :\subseteq X \times \mathbb{N} \to X$ and $g :\subseteq X \times \mathbb{N} \rightrightarrows \mathbb{N}$ are recursive over $S$, then so is $h^\nabla :\subseteq X \times \mathbb{N} \rightrightarrows (X \times \mathbb{N})^\mathbb{N}$, where $h := (f, g) :\subseteq X \times \mathbb{N} \rightrightarrows X \times \mathbb{N}$.*

**Proof.** Let $S$ be a natural structure and let the operations $f :\subseteq X \times \mathbb{N} \to X$ and $g :\subseteq X \times \mathbb{N} \rightrightarrows \mathbb{N}$ be recursive over $S$. We have to show that $h^\nabla$ is recursive over $S$, where $h = (f, g)$. The main idea of the proof is to use sequentialization in order to construct an operation $F :\subseteq X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \rightrightarrows X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N}$ which will be used to iterate $h$ on all possible paths simultaneously (by considering $[F^*]$). In a second step we construct a function $\Sigma :\subseteq (X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N})^\mathbb{N} \to (X \times \mathbb{N})^\mathbb{N}$ which selects a "consistent sequence" from the result of the iteration. We start to construct the operation $F = (A, B, C) :\subseteq X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \rightrightarrows X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N}$ by defining three operations $A$, $B$, $C$. We define $A :\subseteq X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \to X^\mathbb{N}$ by

$$A(\alpha, p, i)\langle \iota, \langle j, k\rangle\rangle := \begin{cases} f(\alpha\langle i, j\rangle, p\langle i, \langle j, k\rangle\rangle) & \text{if } \iota = i + 1 \\ \alpha\langle \iota, \langle j, k\rangle\rangle & \text{otherwise} \end{cases},$$

$B :\subseteq X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ by

$$(B(\alpha, p, i)\langle \iota, \langle\langle j, k\rangle, n\rangle\rangle)_{n\in\mathbb{N}} := \begin{cases} g^\triangle(\alpha\langle i, j\rangle, p\langle i, \langle j, k\rangle\rangle) & \text{if } \iota = i + 1 \\ (p\langle \iota, \langle\langle j, k\rangle, n\rangle\rangle)_{n\in\mathbb{N}} & \text{otherwise} \end{cases}$$

and $C : X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $C(\alpha, p, i) := i + 1$ for all $\alpha \in X^\mathbb{N}$, $p \in \mathbb{N}^\mathbb{N}$ and $\iota, i, j, k \in \mathbb{N}$. The crucial point is the definition of $B$ where the sequentialization scheme has been used. The notation in the definition of $B$ is understood as an intuitive short form of the more precise version

$$\begin{aligned} B(\alpha, p, i) \; := \; \{q \in \mathbb{N}^\mathbb{N} : (\forall j, k)(q\langle i + 1, \langle\langle j, k\rangle, n\rangle\rangle)_{n\in\mathbb{N}} \in g^\triangle(\alpha\langle i, j\rangle, p\langle i, \langle j, k\rangle\rangle) \\ \text{and } (\forall \iota, k)(\iota \neq i + 1 \Longrightarrow q\langle \iota, k\rangle = p\langle \iota, k\rangle)\}. \end{aligned}$$

Now we define $\Sigma :\subseteq (X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N})^\mathbb{N} \to (X \times \mathbb{N})^\mathbb{N}$ by $\Sigma(\alpha_i, p_i, \iota_i)_{i\in\mathbb{N}} := (x_i, n_i)_{i\in\mathbb{N}}$ and

$$\begin{cases} (x_0, n_0) & := & (\alpha_0\langle 0, 0\rangle, p_0\langle 0, \langle 0, 0\rangle\rangle) \\ (x_{i+1}, n_{i+1}) & := & (\alpha_{i+1}\langle i + 1, \langle j, k\rangle\rangle, p_{i+1}\langle i + 1, \langle\langle j, k\rangle, 0\rangle\rangle) \text{ where} \\ & & \langle j, k\rangle := \mu\langle 0, \nu_1, ..., \nu_i\rangle[(\forall \iota \leq i) \; p_{i+1}\langle \iota, \langle 0, \nu_1, ..., \nu_\iota\rangle\rangle = n_\iota] \end{cases} \tag{A.1}$$

for all $(\alpha_i, p_i, \iota_i)_{i\in\mathbb{N}} \in (X^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times \mathbb{N})^\mathbb{N}$ and $\iota \in \mathbb{N}$. Now we define the operation $H :\subseteq X \times \mathbb{N} \rightrightarrows (X \times \mathbb{N})^\mathbb{N}$ by $H(x, n) := \Sigma \circ [F^*]((x, x, x, ...), (n, n, n, ...), 0)$ for all $x \in X$ and $n \in \mathbb{N}$ and we claim $H = h^\nabla$. For the proof, let $x \in X$, $n \in \mathbb{N}$ and let $\alpha := (x, x, x, ...)$ and $p := (n, n, n, ...)$.

"$\subseteq$" Let $(x_i, n_i)_{i\in\mathbb{N}} \in H(x, n)$. Then there exists a sequence $(\alpha_i, p_i, i)_{i\in\mathbb{N}} \in [F^*](\alpha, p, 0)$ such that $(x_i, n_i)_{i\in\mathbb{N}} = \Sigma(\alpha_i, p_i, i)_{i\in\mathbb{N}}$, i.e. such that Equation (A.1) is fulfilled. We have to prove $(x_i, n_i)_{i\in\mathbb{N}} \in h^\nabla(x, n)$. Obviously, $x_0 = \alpha_0\langle 0, 0\rangle = \alpha(0) = x$ and $n_0 = p_0\langle 0, \langle 0, 0\rangle\rangle = p(0) = n$. By induction on $i$ we prove that $(x_{i+1}, n_{i+1}) \in h(x_i, n_i)$, i.e. $x_{i+1} = f(x_i, n_i)$ and $n_{i+1} \in g(x_i, n_i)$ for all $i \in \mathbb{N}$. In case of $i = 0$ the only possible $\langle j, k\rangle$ which fulfills Equation (A.1) is $\langle j, k\rangle =$

$\langle 0,0\rangle = 0$. And actually, $p_1\langle 0,0\rangle = p_0\langle 0,0\rangle = n_0$ by definition of $B$. Thus, we obtain $x_1 = \alpha_1\langle 1,\langle 0,0\rangle\rangle = A(\alpha_0,p_0,0)\langle 1,\langle 0,0\rangle\rangle = f(\alpha_0\langle 0,0\rangle, p_0\langle 0,\langle 0,0\rangle\rangle) = f(x_0,n_0)$ and similarly $n_1 = p_1\langle 1,\langle\langle 0,0\rangle,0\rangle\rangle \in B(\alpha_0,p_0,0)\langle 1,\langle\langle 0,0\rangle,0\rangle\rangle = g(\alpha_0\langle 0,0\rangle, p_0\langle 0,\langle 0,0\rangle\rangle) = g(x_0,n_0)$. Now let us assume that $x_i = f(x_{i-1},n_{i-1})$ and $n_i \in g(x_{i-1},n_{i-1})$ for some fixed $i > 0$. We will show that the same holds for $i+1$. Since Equation (A.1) holds for $i+1$, there is some smallest $\langle j,k\rangle = \langle 0,\nu_1,...,\nu_i\rangle$ such that $p_{i+1}\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = n_\iota$ for all $\iota \leq i$. By definition of $B$ this especially implies $p_i\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = p_{i+1}\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = n_\iota$ for all $\iota \leq i$. Moreover, $j = \langle 0,\nu_1,...,\nu_{i-1}\rangle$ is the smallest number such that $p_i\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = n_\iota$ for all $\iota \leq i-1$. Thus, by induction hypothesis we obtain $\alpha_i\langle i,j\rangle = x_i$ and thus

$$\begin{aligned} x_{i+1} &= \alpha_{i+1}\langle i+1,\langle j,k\rangle\rangle \\ &= A(\alpha_i,p_i,i)\langle i+1,\langle j,k\rangle\rangle \\ &= f(\alpha_i\langle i,j\rangle, p_i\langle i,\langle j,k\rangle\rangle) \\ &= f(x_i,n_i) \end{aligned}$$

and similarly

$$\begin{aligned} n_{i+1} &= p_{i+1}\langle i+1,\langle\langle j,k\rangle,0\rangle\rangle \\ &\in B(\alpha_i,p_i,i)\langle i+1,\langle\langle j,k\rangle,0\rangle\rangle \\ &= g(\alpha_i\langle i,j\rangle, p_i\langle i,\langle j,k\rangle\rangle) \\ &= g(x_i,n_i). \end{aligned}$$

"$\supseteq$" Let $(x_i,n_i)_{i\in\mathbb{N}} \in h^\nabla(x,n)$. Then $x_0 = x$, $n_0 = n$ and $x_{i+1} = f(x_i,n_i)$, $n_{i+1} \in g(x_i,n_i)$ for all $i \in \mathbb{N}$. First we mention that $(x,n) \in \mathrm{dom}(h^\nabla)$ implies $(x,n) \in \mathrm{dom}((f,g)^i)$ for all $i \in \mathbb{N}$ and it is easy to see that this in turn implies $(\alpha,p,0) \in \mathrm{dom}[F^*]$. By induction over $i \in \mathbb{N}$ we will prove that there exists some $(\alpha_i,p_i,i) \in F^*(\alpha,p,0,i)$ for each $i \in \mathbb{N}$ such that Equation (A.1) holds. This implies $(\alpha_i,p_i,i)_{i\in\mathbb{N}} \in [F^*](\alpha,p,0)$ since $(\alpha,p,0) \in \mathrm{dom}[F^*]$ and thus $(x_i,n_i)_{i\in\mathbb{N}} \in H(x,n) = \Sigma \circ [F^*](\alpha,p,0)$. For $i = 0$ we obtain $F^*(\alpha,p,0,0) = (\alpha,p,0)$ and thus $\alpha_0 := \alpha$ and $p_0 := p$ are appropriate choices which fulfill Equation (A.1) since $(x_0,n_0) = (\alpha(0),p(0)) = (\alpha_0\langle 0,0\rangle, p_0\langle 0,\langle 0,0\rangle\rangle)$. Now let us fix $i \in \mathbb{N}$ and let us assume that there exists some $(\alpha_i,p_i,i) \in F^*(\alpha,p,0,i)$ such that Equation (A.1) holds. We have to prove the same for $i+1$. We define $\alpha_{i+1} := A(\alpha_i,p_i,i)$ and we describe the choice of $p_{i+1}$. Let us first consider the case $i > 0$. Since Equation (A.1) holds for $i$, there is some smallest number $\langle j,k\rangle = \langle 0,\nu_1,...,\nu_{i-1}\rangle$ such that $p_i\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = n_\iota$ for all $\iota \leq i-1$. It follows $p_i\langle i,\langle\langle j,k\rangle,0\rangle\rangle = n_i$ and $\alpha_i\langle i,\langle j,k\rangle\rangle = x_i$. Let $\nu_i := 0$. Then $\langle 0,\nu_1,...,\nu_i\rangle = \langle\langle j,k\rangle,0\rangle$ and thus $p_i\langle \iota,\langle 0,\nu_1,...,\nu_\iota\rangle\rangle = n_\iota$ for all $\iota \leq i$ and $\langle 0,\nu_1,...,\nu_i\rangle$ is even the minimal number with this property. Let $\langle j',k'\rangle := \langle 0,\nu_1,...,\nu_i\rangle = \langle\langle j,k\rangle,0\rangle$.

In case of $i = 0$ we choose $\langle j', k' \rangle := \langle 0, 0 \rangle = 0$. In both cases we obtain $p_i\langle i, \langle j', k' \rangle\rangle = n_i$ and $\alpha_i\langle i, j' \rangle = x_i$. Since $n_{i+1} \in g(x_i, n_i)$, there is some sequence $q \in g^{\triangle}(x_i, n_i) = g^{\triangle}(\alpha_i\langle i, j' \rangle, p_i\langle i, \langle j', k' \rangle\rangle)$ such that $q(0) = n_{i+1}$. Thus, there is some $p_{i+1} \in B(\alpha_i, p_i, i)$ such that $p_{i+1}\langle i+1, \langle\langle j', k' \rangle, 0\rangle\rangle = n_{i+1}$. Moreover, we obtain $\alpha_{i+1}\langle i+1, \langle j', k' \rangle\rangle = f(\alpha_i\langle i, j' \rangle, p_i\langle i, \langle j', k' \rangle\rangle) = f(x_i, n_i) = x_{i+1}$. Since $p_{i+1}\langle \iota, \kappa \rangle = p_i\langle \iota, \kappa \rangle$ for all $\iota \leq i$ and $\kappa \in \mathbb{N}$ by definition of $B$, it follows that $\alpha_{i+1}, p_{i+1}$ fulfill Equation (A.1).

It remains to prove that $H$ is recursive over $S$. Therefore, it suffices to show that $A, B, C$ and $\Sigma$ are recursive over $S$, since then $F$ and thus $H$ are recursive over $S$ as well. $C$ is obviously recursive over $S$ and it is easy to see that $A$, $B$ and $\Sigma$ can be constructed from recursive operations over $S$ with the help of evaluation, transposition, definition by cases and primitive recursion (see Propositions 2.5.8 and 2.5.4 in [Bra99a] for the previous two closure schemes). We sketch the construction of $B$ a bit more detailed. Therefore, we define operations $G :\subseteq X^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightrightarrows \mathbb{N}^{\mathbb{N}}$, $I : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$, $D :\subseteq X^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and $\Pi : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $G(\alpha, p, i, j, k) := g^{\triangle}(\alpha\langle i, j \rangle, p\langle i, \langle j, k \rangle\rangle)$, $I(p, \iota, j, k)(n) := p\langle \iota, \langle\langle j, k \rangle, n\rangle\rangle$,

$$D(\alpha, p, i, \langle \iota, j, k \rangle) := \begin{cases} G(\alpha, p, i, j, k) & \text{if } \iota = i+1 \\ I(p, \iota, j, k) & \text{else} \end{cases}$$

and $\Pi(P)\langle \iota, \langle\langle j, k \rangle, n\rangle\rangle := P\langle \iota, j, k \rangle(n)$. By evaluation and sequentialization it is easy to see that $G$ is recursive over $S$, by evaluation and transposition it follows that $I$ is recursive over $S$. By definition by cases one obtains that $D$ is recursive over $S$ and, finally, by evaluation and transposition it follows that $\Pi$ is recursive over $S$. We obtain

$$\begin{aligned} q \in B(\alpha, p, i) \quad &\iff \quad (\forall j, k)(q\langle i+1, \langle\langle j, k \rangle, n\rangle\rangle)_{n \in \mathbb{N}} \in G(\alpha, p, i, j, k) \\ &\qquad\quad \text{and } (\forall \iota \neq i+1, j, k)(q\langle \iota, \langle\langle j, k \rangle, n\rangle\rangle)_{n \in \mathbb{N}} = I(p, \iota, j, k) \\ &\iff \quad (\forall \iota, j, k)(q\langle \iota, \langle\langle j, k \rangle, n\rangle\rangle)_{n \in \mathbb{N}} \in D(\alpha, p, i, \langle \iota, j, k \rangle) \\ &\iff \quad q \in \Pi \circ [D](\alpha, p, i) \end{aligned}$$

for all $\alpha \in X^{\mathbb{N}}$, $p \in \mathbb{N}^{\mathbb{N}}$ and $i \in \mathbb{N}$. Thus, $B = \Pi \circ [D]$ is recursive over $S$ as well. $\square$

The previous proposition has been applied in the proof of the Quasi-Metric Structure Theorem 4.9.1. If one would add the sequential iteration scheme to the recursive closure schemes, then all the results would remain the same (since by Proposition A.2 sequential iteration is an "admissible" closure scheme). In this case the proof of Theorem 4.9.1 would not require the previous proposition. However, this modified setting of recursive closure schemes might be slightly more powerful (in the sense that the class of perfect structures could be larger) and consequently the presented result is formally stronger.

# Appendix B

# Effective Subsets of Metric Spaces

In this appendix we briefly summarize some notions of effectivity for closed subsets of computable metric spaces (as they have been introduced and studied in [BW99, BPar, Bra02b]). We start with the notion of r.e., strongly co-r.e. and strongly recursive sets.

**Definition B.0.13 (R.e. closed subsets)** Let $(X, d, \alpha)$ be a recursive metric space and let $A \subseteq X$ be a closed subset.

(1) $A$ is called *r.e. closed*, if $\{\langle n, k\rangle \in \mathbb{N} : A \cap B(\alpha(n), \overline{k}) \neq \emptyset\}$ is r.e.

(2) $A$ is called *strongly co-r.e. closed*, if $\{\langle n, k\rangle \in \mathbb{N} : A \cap \overline{B}(\alpha(n), \overline{k}) = \emptyset\}$ is r.e.

(3) $A$ is called *strongly recursive closed*, if $A$ is r.e. closed as well as strongly co-r.e. closed.

These notions are closely related to the following ones, which can be defined via the distance function. The distance function can be considered as a continuous substitute of the characteristic function.

**Definition B.0.14 (Located subsets)** Let $X$ be a recursive metric space and let $A \subseteq X$.

(1) $A$ is called *lower semi-located*, if $d_A : X \to \mathbb{R}$ is lower semi-computable,

(2) $A$ is called *upper semi-located*, if $d_A : X \to \mathbb{R}$ is upper semi-computable,

(3) $A$ is called *located*, if $d_A : X \to \mathbb{R}$ is computable.

Next, we define the notion of a co-r.e. subset (which are also known as $\Pi^0_1$–subsets). In the context of computable analysis, these subsets have already been studied by Lacombe [Lac57].

**Definition B.0.15 (Co-r.e. closed subsets)** Let $(X, d, \alpha)$ be a recursive metric space and let $A \subseteq X$ be a closed subset. Then $A$ is called *co-r.e. closed*, if there exist computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that

$$X \setminus A = \bigcup_{i=0}^{\infty} B(\alpha f(i), \overline{g(i)}).$$

Moreover, $A$ is called *recursive closed*, if $A$ is r.e. closed as well as co-r.e. closed.

As a final definition we mention the notion of effective separability.

**Definition B.0.16 (Effective separability)** Let $(X, d, \alpha)$ be a computable metric space and let $A \subseteq X$ be a subset. Then $A$ is called *effectively separable*, if there exists a computable sequence $f : \mathbb{N} \to X$ such that $\overline{\mathrm{range}(f)} = A$.

The effectivity notions introduced in this appendix are directly related to the representations introduced in Definition 5.1.6. More precisely, a non-empty closed subset $A \subseteq X$ is r.e. closed, strongly co-r.e. closed, strongly recursive closed, lower semi-located, upper semi-located, located, co-r.e. or effectively separable, if and only if it is $\delta^<$–, $\delta^>$–, $\delta^=$–, $\delta^>_{\mathrm{dist}}$–, $\delta^<_{\mathrm{dist}}$–, $\delta^=_{\mathrm{dist}}$–, $\delta_{\mathrm{union}}$– or $\delta_{\mathrm{range}}$–computable. Analogously to Figure 5.1 we obtain the logical relation of notions of effectivity for closed subsets of recursive metric spaces displayed in Figure B.1.

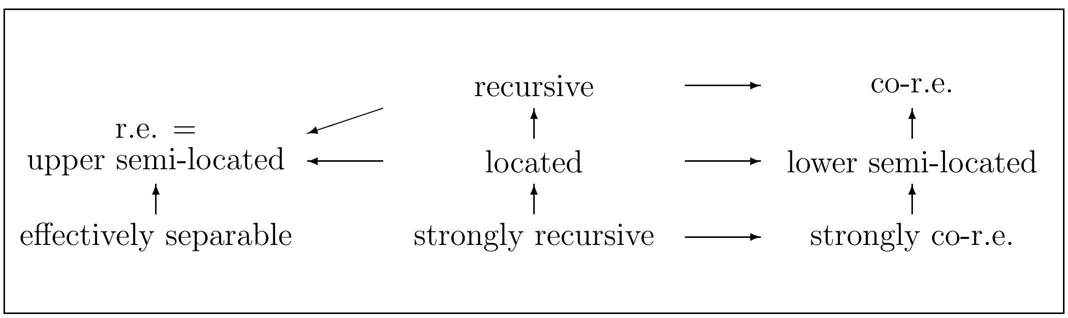


Figure B.1: Notions of effectivity for non-empty closed subsets of metric spaces

These logical relations have been established in [BPar, Bra02b]. In case that $X$ is a nice recursively locally compact recursive metric space, it follows from Proposition 5.1.7 that all vertical arrows can be inverted.

# Tables of Perfect Structures

In the following we will list some perfect (pre)structures which have been investigated in the previous chapters with the precise definition of their initial operations. Additionally, the tables include the recursive points and the underlying topologies of the corresponding structures. We recall the fact that predicates (sets) are identified with their semi-characteristic operation.

| $\mathbb{N}$ | **Naturals** | $\{0, 1, 2, ...\}$, discrete topology |
|---|---|---|
| $0$ | constant | $0$ |
| $n$ | identity | $\mathrm{id} : \mathbb{N} \to \mathbb{N}, n \mapsto n$ |
| $n + 1$ | successor function | $s : \mathbb{N} \to \mathbb{N}, n \mapsto n + 1$ |

| $\mathbb{Q}$ | **Rationals** | rational numbers, discrete topology |
|---|---|---|
| $0$ | constant | $0$ |
| $1$ | constant | $1$ |
| $x + y$ | addition | $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, (x, y) \mapsto x + y$ |
| $-x$ | negation | $\mathbb{Q} \to \mathbb{Q}, x \mapsto -x$ |
| $x \cdot y$ | multiplication | $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, (x, y) \mapsto x \cdot y$ |
| $1/x$ | inversion | $\subseteq \mathbb{Q} \to \mathbb{Q}, x \mapsto 1/x$ |
| $=$ | equality test | $\{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}$ |

| $\mathbb{R}$ | **Reals** | computable real numbers, Euclidean topology |
|---|---|---|
| $0$ | constant | $0$ |
| $1$ | constant | $1$ |
| $x+y$ | addition | $\mathbb{R}\times\mathbb{R}\to\mathbb{R}, (x,y)\mapsto x+y$ |
| $-x$ | negation | $\mathbb{R}\to\mathbb{R}, x\mapsto -x$ |
| $x\cdot y$ | multiplication | $\mathbb{R}\times\mathbb{R}\to\mathbb{R}, (x,y)\mapsto x\cdot y$ |
| $1/x$ | inversion | $\subseteq\mathbb{R}\to\mathbb{R}, x\mapsto 1/x$ |
| Lim | limit | $\mathrm{Lim} :\subseteq \mathbb{R}^{\mathbb{N}}\to\mathbb{R}, (x_n)_{n\in\mathbb{N}}\mapsto \lim_{n\to\infty} x_n$ |
| | | $\mathrm{dom}(\mathrm{Lim}) := \{(x_n)_{n\in\mathbb{N}} : (\forall n>k)\ \lvert x_n - x_k\rvert \le 2^{-k}\}$ |
| $x<y$ | comparison | $\{(x,y)\in\mathbb{R}\times\mathbb{R} : x<y\}$ |

| $\mathbb{R}_<$ | **Reals** | left computable real numbers,<br>lower Euclidean topology |
|---|---|---|
| $x$ | identity | $\mathbb{R}_<\to\mathbb{R}_<, x\mapsto x$ |
| Sup | supremum | $\mathbb{R}^{\mathbb{N}}\to\mathbb{R}_<, (y_n)_{n\in\mathbb{N}}\mapsto \sup_{n\in\mathbb{N}} y_n$ |
| $x>y$ | comparison | $\{(x,y)\in\mathbb{R}_<\times\mathbb{R} : x>y\}$ |

| $\mathbb{R}_>$ | **Reals** | right computable real numbers,<br>upper Euclidean topology |
|---|---|---|
| $x$ | identity | $\mathbb{R}_>\to\mathbb{R}_>, x\mapsto x$ |
| Inf | infimum | $\mathbb{R}^{\mathbb{N}}\to\mathbb{R}_>, (y_n)_{n\in\mathbb{N}}\mapsto \inf_{n\in\mathbb{N}} y_n$ |
| $x<y$ | comparison | $\{(x,y)\in\mathbb{R}_>\times\mathbb{R} : x<y\}$ |

| $\mathcal{K}(X)$ | **Compact subsets** | recursive compact sets, Vietoris topology |
|---|---|---|
| $\{x\}$ | injection | $X \to \mathcal{K}(X), x \mapsto \{x\}$ |
| $A$ | identity | $\mathcal{K}(X) \to \mathcal{K}(X), A \mapsto A$ |
| $A \cup B$ | union | $\mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X), (A, B) \mapsto A \cup B$ |
| $d_{\mathcal{K}}$ | Hausdorff metric | $d_{\mathcal{K}} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}$, |
| | | $(A, B) \mapsto \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$ |
| Lim | limit | $\mathrm{Lim} :\subseteq \mathcal{K}(X)^{\mathbb{N}} \to \mathcal{K}(X), (A_n)_{n \in \mathbb{N}} \mapsto \lim_{n \to \infty} A_n$ |
| | | $\mathrm{dom}(\mathrm{Lim}) := \{(A_n) : (\forall n > k)\ d_{\mathcal{K}}(A_n, A_k) \leq 2^{-k}\}$ |
| Here, $(X, d)$ is a recursive metric space. | | |

| $\mathcal{K}_>(X)$ | **Compact subsets** | co-r.e. compact sets, upper Vietoris topology |
|---|---|---|
| $A$ | identity | $\mathcal{K}_>(X) \to \mathcal{K}_>(X), A \mapsto A$ |
| $d'_{\mathcal{K}}$ | quasi-metric | $d'_{\mathcal{K}} : \mathcal{K}_>(X) \times \mathcal{K}(X) \to \mathbb{R}_>$, |
| | | $(A, B) \mapsto \sup_{a \in A} \inf_{b \in B} d(a, b)$ |
| $\bigcap_{n=0}^{\infty} B_n$ | intersection | $\subseteq \mathcal{K}(X)^{\mathbb{N}} \to \mathcal{K}_>(X), (B_n)_{n \in \mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} B_n$ |
| Here, $(X, d)$ is a complete recursive metric space. | | |

| $\mathcal{K}_<(X)$ | **Compact subsets** | r.e. compact sets, lower Vietoris topology |
|---|---|---|
| $A$ | identity | $\mathcal{K}_<(X) \to \mathcal{K}_<(X), A \mapsto A$ |
| $\overline{d'_{\mathcal{K}}}$ | conj. quasi-metric | $\overline{d'_{\mathcal{K}}} : \mathcal{K}_<(X) \times \mathcal{K}(X) \to \mathbb{R}_>$, |
| | | $(A, B) \mapsto \sup_{b \in B} \inf_{a \in A} d(a, b)$ |
| $\overline{\bigcup_{n=0}^{\infty} B_n}$ | closed union | $\subseteq \mathcal{K}(X)^{\mathbb{N}} \to \mathcal{K}_<(X), (B_n)_{n \in \mathbb{N}} \mapsto \overline{\bigcup_{n=0}^{\infty} B_n}$ |
| Here, $(X, d)$ is a complete recursive metric space. | | |

| $\mathcal{A}(X)$ | **Closed subsets** | recursive closed sets, Fell topology |
|---|---|---|
| $\{x\}$ | injection | $X \to \mathcal{A}(X), x \mapsto \{x\}$ |
| $A$ | identity | $\mathcal{A}(X) \to \mathcal{A}(X), A \mapsto A$ |
| $A \cup B$ | union | $\mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto A \cup B$ |
| $d_{\mathcal{A}}$ | metric | $d_{\mathcal{A}} : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathbb{R}$, |
| | | $(A, B) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \lvert d_A - d_B \rvert_{K_i}$ |
| Lim | limit | $\mathrm{Lim} :\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}(X), (A_n)_{n\in\mathbb{N}} \mapsto \lim\limits_{n\to\infty} A_n$ |
| | | $\mathrm{dom}(\mathrm{Lim}) := \{(A_n) : (\forall n > k)\ d_{\mathcal{A}}(A_n, A_k) \leq 2^{-k}\}$ |

Here, $(X, d)$ is a nice recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

| $\mathcal{A}_>(X)$ | **Closed subsets** | co-r.e. closed sets, upper Fell topology |
|---|---|---|
| $A$ | identity | $\mathcal{A}_>(X) \to \mathcal{A}_>(X), A \mapsto A$ |
| $d'_{\mathcal{A}}$ | quasi-metric | $d'_{\mathcal{A}} : \mathcal{A}_>(X) \times \mathcal{A}(X) \to \mathbb{R}_>$, |
| | | $(A, B) \mapsto \sum\limits_{i=0}^{\infty} 2^{-i-1} \lvert d_B \dot{-} d_A \rvert_{K_i}$ |
| $\bigcap\limits_{n=0}^{\infty} B_n$ | intersection | $\subseteq \mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}_>(X), (B_n)_{n\in\mathbb{N}} \mapsto \bigcap\limits_{n=0}^{\infty} B_n$ |

Here, $(X, d)$ is a nice recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

| $\mathcal{A}_<(X)$ | **Closed subsets** | r.e. closed sets, lower Fell topology |
|---|---|---|
| $A$ | identity | $\mathcal{A}_<(X) \to \mathcal{A}_<(X), A \mapsto A$ |
| $\overline{d'_{\mathcal{A}}}$ | conj. quasi-metric | $\overline{d'_{\mathcal{A}}} : \mathcal{A}_<(X) \times \mathcal{A}(X) \to \mathbb{R}_>$, |
| | | $(A, B) \mapsto \sum\limits_{i=0}^{\infty} 2^{-i-1} \lvert d_A \dot{-} d_B \rvert_{K_i}$ |
| $\overline{\bigcup\limits_{n=0}^{\infty} B_n}$ | closed union | $\mathcal{A}(X)^{\mathbb{N}} \to \mathcal{A}_<(X), (B_n)_{n\in\mathbb{N}} \mapsto \overline{\bigcup\limits_{n=0}^{\infty} B_n}$ |

Here, $(X, d)$ is a nice recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

| $\mathcal{C}(X)$ | **Continuous functions** | computable functions, compact open topology |
|---|---|---|
| 1 | constant function | $\{()\} \to \mathcal{C}(X), () \mapsto \hat{1}$ |
| | | $\hat{1} : X \to \mathbb{R}, x \mapsto 1$ |
| $d_x$ | point distance | $X \to \mathcal{C}(X), x \mapsto d_x$ |
| | | $d_x : X \to \mathbb{R}, y \mapsto d(x,y)$ |
| $f$ | identity | $\mathcal{C}(X) \to \mathcal{C}(X), f \mapsto f$ |
| $y \cdot f$ | scalar product | $\mathbb{R} \times \mathcal{C}(X) \to \mathcal{C}(X), (y,f) \mapsto y \cdot f$ |
| $f + g$ | addition | $\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X), (f,g) \mapsto f + g$ |
| $f \cdot g$ | multiplication | $\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X), (f,g) \mapsto f \cdot g$ |
| $d_{\mathcal{C}}$ | metric | $d_{\mathcal{C}} : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R},$ |
| | | $(f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{\lvert f-g\rvert_{K_i}}{1+\lvert f-g\rvert_{K_i}}$ |
| Lim | limit | $\mathrm{Lim} :\subseteq \mathcal{C}(X)^{\mathbb{N}} \to \mathcal{C}(X), (f_n)_{n\in\mathbb{N}} \mapsto \lim\limits_{n\to\infty} f_n$ |
| | | $\mathrm{dom}(\mathrm{Lim}) := \{(f_n) : (\forall n > k) d_{\mathcal{C}}(f_n, f_k) \le 2^{-k}\}$ |

Here, $(X,d)$ is a recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

| $\mathcal{USC}(X)$ | **Upper semi-continuous fcts.** | upper semi-computable functions, upper compact open topology |
|---|---|---|
| $f$ | identity | $\mathcal{USC}(X) \to \mathcal{USC}(X), f \mapsto f$ |
| $d_{\mathcal{USC}}$ | quasi-metric | $d_{\mathcal{USC}} : \mathcal{USC}(X) \times \mathcal{C}(X) \to \mathbb{R}_>,$ |
| | | $(f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{\lvert f \dot{-} g\rvert_{K_i}}{1+\lvert f \dot{-} g\rvert_{K_i}}$ |
| Inf | infimum | $\mathrm{Inf} :\subseteq \mathcal{C}(X)^{\mathbb{N}} \to \mathcal{USC}(X), (g_n)_{n\in\mathbb{N}} \mapsto \inf\limits_{n\in\mathbb{N}} g_n$ |

Here, $(X,d)$ is a recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

| | | |
|---|---|---|
| $\mathcal{LSC}(X)$ | **Lower semi-continuous fcts.** | lower semi-computable functions, lower compact open topology |
| $f$ | identity | $\mathcal{LSC}(X) \to \mathcal{LSC}(X), f \mapsto f$ |
| $d_{\mathcal{LSC}}$ | conj. quasi-metric | $d_{\mathcal{LSC}} : \mathcal{LSC}(X) \times \mathcal{C}(X) \to \mathbb{R}_>$, $(f,g) \mapsto \sum_{i=0}^{\infty} 2^{-i-1} \frac{\|g \dot{-} f\|_{K_i}}{1+\|g \dot{-} f\|_{K_i}}$ |
| Sup | supremum | $\mathrm{Sup} :\subseteq \mathcal{C}(X)^{\mathbb{N}} \to \mathcal{LSC}(X), (g_n)_{n\in\mathbb{N}} \mapsto \sup_{n\in\mathbb{N}} g_n$ |

Here, $(X,d)$ is a recursively locally compact recursive metric space with recursive exhausting sequence $(K_i)_{i\in\mathbb{N}}$.

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# Symbols

| | |
|---|---|
| $\emptyset$ | Empty set |
| $\mathbb{N}$ | Set of natural numbers $\{0, 1, 2, ...\}$ |
| $\mathbb{Z}$ | Set of integer numbers |
| $\mathbb{Q}$ | Set of rational numbers |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{N}^{\mathbb{N}}$ | Set of sequences of natural numbers |
| | |
| $(x, y)$ | Open interval of real numbers |
| $(x, y]$ | Left open interval of real numbers |
| $[x, y)$ | Right open interval of real numbers |
| $[x, y]$ | Closed interval of real numbers |
| | |
| $\infty$ | Infinity |
| | |
| $B_<(x, \varepsilon)$ | Open ball of quasi-metric space with left center $x$ |
| $B_>(x, \varepsilon)$ | Open ball of quasi-metric space with right center $x$ |
| $B(x, \varepsilon)$ | Open ball of metric space with center $x$ and radius $\varepsilon$ |
| $\overline{B}(x, \varepsilon)$ | Closed ball of metric space with center $x$ and radius $\varepsilon$ |
| $\tau_<$ | Lower topology induced by balls $B_<(x, \varepsilon)$ |
| $\tau_>$ | Upper topology induced by balls $B_>(x, \varepsilon)$ |
| $\tau \sqcap \tau'$ | Join of topologies |
| $\overline{d}$ | Conjugate quasi-metric associated with quasi-metric $d$ |
| $d_\star$ | Metric associated with quasi-metric $d$ |
| $X_<$ | Space $X$ endowed with lower topology |
| $X_>$ | Space $X$ endowed with (weak) upper topology |
| | |
| $\sqsubseteq$ | Partial order |
| max | Maximum |
| min | Minimum |
| $\sup = \sqcup$ | Supremum |
| $\inf = \sqcap$ | Infimum |

| | |
|---|---|
| Sup | Supremum operator |
| Inf | Infimum operator |
| $\overline{A}$ | Closure of set $A$ |
| $A^\circ$ | Interior of set $A$ |
| $\partial A$ | Border of set $A$ |
| $A^+$ | Miss set of $A$ |
| $A^-$ | Hit set of $A$ |
| $\lvert x \rvert$ | Absolute value of real number $x$ |
| $\lVert\ \rVert$ | Norm |
| $\lVert\ \rVert_K$ | Supremum of norm over set $K$ |
| $\mathcal{K}(X) = \mathcal{K}$ | Set of non-empty compact subsets of $X$ |
| $\mathcal{A}(X) = \mathcal{A}$ | Set of non-empty closed subsets of $X$ |
| $\mathcal{C}(X)$ | Set of continuous functions $f : X \to \mathbb{R}$ |
| $\mathcal{C}(X, Y)$ | Set of continuous functions $f : X \to Y$ |
| $\mathcal{LSC}(X)$ | Set of lower semi-continuous functions $f : X \to \mathbb{R}$ |
| $\mathcal{USC}(X)$ | Set of upper semi-continuous functions $f : X \to \mathbb{R}$ |
| $d_{\mathbb{R}^n}$ | Euclidean metric |
| $d'_{\mathcal{K}}$ | Quasi-metric on $\mathcal{K}(X)$ (excess of $A$ over $B$) |
| $d_{\mathcal{K}}$ | Hausdorff metric on $\mathcal{K}(X)$ |
| $d'_{\mathcal{A}}$ | Quasi-metric on $\mathcal{A}(X)$ |
| $d_{\mathcal{A}}$ | Metric on $\mathcal{A}(X)$ |
| $d_{\mathcal{LSC}}$ | Quasi-metric on $\mathcal{LSC}(X)$ |
| $d_{\mathcal{USC}}$ | Quasi-metric on $\mathcal{USC}(X)$ |
| $d_{\mathcal{C}}$ | Metric on $\mathcal{C}(X)$ |
| $\mathbb{R}_<$ | $\mathbb{R}$ with lower topology |
| $\mathbb{R}_>$ | $\mathbb{R}$ with upper topology |
| $\widehat{\mathbb{R}}_<$ | Extended reals with lower topology |
| $\widehat{\mathbb{R}}_>$ | Extended reals with upper topology |
| $\mathcal{K}_<(X)$ | $\mathcal{K}(X)$ with lower Vietoris topology |
| $\mathcal{K}_>(X)$ | $\mathcal{K}(X)$ with upper Vietoris topology |
| $\mathcal{A}_<(X)$ | $\mathcal{A}(X)$ with lower Fell topology |
| $\mathcal{A}_>(X)$ | $\mathcal{A}(X)$ with upper Fell topology |

| | |
|---|---|
| $\alpha_X$ | Dense sequence in $X$ |
| Lim | Limit operator for rapidly converging Cauchy sequences |
| dist | Distance operator |
| $d_A$ | Distance function of set $A$ |
| $d_x$ | Distance function of point $x$ |
| | |
| $x \in X$ | $x$ is element of $X$ |
| $X \subseteq Y$ | $X$ is included in $Y$ |
| $X^0$ | Set of empty tuple $\{()\}$ |
| $X^c$ | Complement of set $X$ |
| $X \cup Y$ | Union of set $X$ and $Y$ |
| $X \cap Y$ | Intersection of set $X$ and $Y$ |
| $X \setminus Y$ | Difference of set $X$ and $Y$ |
| $X \times Y$ | Product of set $X$ and $Y$ |
| $2^X$ | Power set of set $X$ |
| $X^{\mathbb{N}}$ | Set of sequences $s : \mathbb{N} \to X$ |
| $Y^X$ | Set of functions $f : X \to Y$ |
| | |
| $f :\subseteq X \to Y$ | Partial function |
| $f : X \to Y$ | Total function |
| $f :\subseteq X \rightrightarrows Y$ | Partial operation (many-valued function) |
| $f : X \rightrightarrows Y$ | Total operation (many-valued function) |
| $f^{-1}$ | Inverse operation of $f$ |
| $f(x) = \uparrow$ | Operation $f$ is undefined at $x$ |
| $f(X)$ | Image of set $X$ under operation $f$ |
| $f^{-1}(Y)$ | Preimage of set $Y$ under operation $f$ |
| $f\|_X$ | Restriction of operation $f$ in the domain |
| $f\|^Y$ | Restriction of operation $f$ in the range |
| | |
| $\mathrm{dom}(f)$ | Domain of operation $f$ |
| $\mathrm{range}(f)$ | Range of operation $f$ |
| $\mathrm{graph}(f)$ | Graph of operation $f$ |
| $\mathrm{hypo}(f)$ | Hypograph of function $f : X \to \mathbb{R}$ |
| $\mathrm{epi}(f)$ | Epigraph of function $f : X \to \mathbb{R}$ |
| | |
| $\mathrm{id}_X$ | Identity of set $X$ |
| $\mathrm{cf}_X$ | Characteristic function of $X$ |
| $c_X$ | Semi-characteristic operation of $X$ |
| $\Omega_X$ | Omnipotent operation of $X$ |
| in | Injection which maps a point to a single-valued set |

| | |
|---|---|
| $f_0$ | Section of operation $f$ |
| $f_i$ | Projection of operation $f$ on the $i$-th component |
| $(f, g)$ | Juxtaposition of operation $f$ and $g$ |
| $f \times g$ | Product of operation $f$ and $g$ |
| $f \circ g$ | Composition of operation $f$ and $g$ |
| $f^*$ | Iteration of operation $f$ |
| $f_*$ | Evaluation of operation $f$ |
| $[f]$ | Transposition of operation $f$ |
| $f^{\mathbb{N}}$ | Exponentiation of operation $f$ |
| $f^{\triangle}$ | Sequentialization of operation $f$ |
| $f^{\leftrightarrow}$ | (Twisted) Inversion of operation $f$ |
| $f^{\nabla}$ | Sequential iteration of operation $f$ |
| | |
| $\check{f}$ | Transposition of function $f$ |
| ev | Evaluation function |
| | |
| $\mathbf{N}$ | Structure of natural numbers |
| $\mathbf{R}$ | Structure of real numbers |
| $\mathbf{R}_<$ | Lower structure of real numbers |
| $\mathbf{R}_>$ | Upper structure of real numbers |
| $\mathbf{X}$ | Metric structure |
| $\mathbf{X}_>$ | Quasi-metric structure |
| $\widehat{\mathbf{X}}_>$ | Extended quasi-metric structure |
| $\widehat{\mathbf{R}}_<$ | Extended lower structure of real numbers |
| $\widehat{\mathbf{R}}_>$ | Extended upper structure of real numbers |
| $\mathcal{K}(\mathbf{X})$ | Structure of non-empty compact subsets |
| $\mathcal{K}_<(\mathbf{X})$ | Lower structure of non-empty compact subsets |
| $\mathcal{K}_>(\mathbf{X})$ | Upper structure of non-empty compact subsets |
| $\widehat{\mathcal{K}}_<(\mathbf{X})$ | Extended lower structure of non-empty compact subsets |
| $\mathcal{A}(\mathbf{X})$ | Structure of non-empty closed subsets |
| $\mathcal{A}_<(\mathbf{X})$ | Lower structure of non-empty closed subsets |
| $\mathcal{A}_>(\mathbf{X})$ | Upper structure of non-empty closed subsets |
| $\widehat{\mathcal{A}}_<(\mathbf{X})$ | Extended lower structure of non-empty closed subsets |
| $\mathcal{C}(\mathbf{X})$ | Structure of continuous functions $f : X \to \mathbb{R}$ |
| $\mathcal{LSC}(\mathbf{X})$ | Structure of lower semi-continuous functions $f : X \to \mathbb{R}$ |
| $\mathcal{USC}(\mathbf{X})$ | Structure of upper semi-continuous functions $f : X \to \mathbb{R}$ |
| | |
| $(X; f_1, ..., f_n)$ | (Pre)structure with universe and initial operations |
| $S \sqsubseteq T$ | Substructure |
| $S \oplus T$ | Union of (pre)structures |

| | |
|---|---|
| $\dot{-}$ | Truncated difference |
| $S$ | Successor function |
| $\mathrm{pr}_i^{(n)}, \mathrm{pr}_i$ | Projection of an $n$-tuple to the $i$-th component |
| $\langle n, k\rangle$ | Cantor's pairing function for natural numbers $n, k$ |
| $\langle p_1, ..., p_n\rangle$ | Pairing of sequences $p_1, ..., p_n$ |
| $\langle p_1, p_2, ...\rangle$ | Pairing of infinitely many sequences $p_1, p_2, ...$ |
| $\pi_i$ | Projection of the $i$-th component of the inverse of a pairing function |

| | |
|---|---|
| $\mathbb{N}^*$ | Set of finite words over $\mathbb{N}$ |
| $\sqsubseteq$ | Prefix relation for words and sequences |
| $\mathrm{lg}(w)$ | Length of word $w$ |
| $\nu^*$ | Standard numbering of $\mathbb{N}^*$ |
| $\overline{w}$ | Number of word $w$ with respect to $\nu^*$ |
| $wp$ | Concatenation of word $w$ and sequence $p$ |
| $w\mathbb{N}^{\mathbb{N}}$ | Set of all sequences which extend the word $w$ |
| $\langle p, \mathbb{N}^{\mathbb{N}}\rangle$ | Set of all tuples $\langle p, q\rangle$ for fixed $p$ |
| $\hat{n}$ | Sequence $n00...$ |
| $\overline{n}$ | Rational number $\alpha_{\mathbb{R}}(n)$ |

| | |
|---|---|
| $\delta_X$ | (Cauchy) representation of (metric space) $X$ |
| $\delta_{X_<}$ | Lower Dedekind representation of quasi-metric space $X$ |
| $\delta_{X_>}$ | Upper Dedekind representation of quasi-metric space $X$ |
| $\delta_{\mathcal{LSC}(X)}$ | Upper Dedekind representation $\delta_{\mathcal{LSC}(X)_>}$ of $\mathcal{LSC}(X)$ |
| $\delta_{\mathcal{USC}(X)}$ | Upper Dedekind representation $\delta_{\mathcal{USC}(X)_>}$ of $\mathcal{USC}(X)$ |
| $\delta_{\mathcal{A}_<(X)}$ | Lower Dedekind representation $\delta_{\mathcal{A}(X)_<}$ of $\mathcal{A}(X)$ |
| $\delta_{\mathcal{A}_>(X)}$ | Upper Dedekind representation $\delta_{\mathcal{A}(X)_>}$ of $\mathcal{A}(X)$ |
| $[\delta_1, ..., \delta_n]$ | Product representation |
| $\delta^\infty$ | Sequence representation |
| $[\delta \to \delta']$ | Function space representation |
| $\delta \le \delta'$ | Reducibility of representations |
| $\delta \equiv \delta'$ | Equivalence of representations |
| $\delta \le_{\mathrm{t}} \delta'$ | Topological reducibility of representations |
| $\delta \equiv_{\mathrm{t}} \delta'$ | Topological equivalence of representations |
| $\delta \sqcap \delta'$ | Conjunction of representations |

| | |
|---|---|
| $\delta^<, \delta^>, \delta^=$ | Positive, negative and symmetric representations of $\mathcal{A}(X)$ |
| $\delta^<_{\mathrm{dist}}, \delta^>_{\mathrm{dist}}, \delta^=_{\mathrm{dist}}$ | Representations of $\mathcal{A}(X)$ via distance functions |
| $\delta_{\mathrm{range}}, \delta_{\mathrm{union}}$ | Representations of $\mathcal{A}(X)$ via sequences and unions of balls |

| | |
|---|---|
| $\eta^{\omega\omega}, \eta$ | Representation of continuous functions $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $G_\delta$–domain |
| $\delta_B$ | Standard representation of second-countable $T_0$–space with subbase numbering $B$ |
| $\sqsubseteq_n$ | Partial orders associated with second-countable $T_0$–spaces |
| $\equiv_n$ | Corresponding equivalence relations |

# Index