Computable Versions of Baire's Category Theorem

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Abstract. We study different computable versions of Baire's Category Theorem in computable analysis. Similarly, as in constructive analysis, different logical forms of this theorem lead to different computational interpretations. We demonstrate that, analogously to the classical theorem, one of the computable versions of the theorem can be used to construct interesting counterexamples, such as a computable but nowhere differentiable function.

Keywords: computable analysis, functional analysis.

1 Introduction

Baire's Category Theorem states that a complete metric space X cannot be decomposed into a countable union of nowhere dense closed subsets A_n (cf. [7]). Classically, we can bring this statement into the following two equivalent logical forms:

1. For all sequences $(A_n)_{n\in\mathbb{N}}$ of closed and nowhere dense subsets $A_n \subseteq X$, there exists some point $x \in X \setminus \bigcup_{n=0}^{\infty} A_n$,
2. for all sequences $(A_n)_{n\in\mathbb{N}}$ of closed subsets $A_n \subseteq X$ with $X = \bigcup_{n=0}^{\infty} A_n$, there exists some $k \in \mathbb{N}$ such that A_k is somewhere dense.

Both logical forms of the classical theorem have interesting applications. While the first version is often used to ensure the existence of certain types of counterexamples, the second version is for instance used to prove some important theorems in functional analysis, like the Open Mapping Theorem and the Closed Graph Theorem [7]. However, from the computational point of view the content of both logical forms of the theorem is different. This has already been observed in constructive analysis, where a discussion of the theorem can be found in [6].

We will study the theorem from the point of view of computable analysis, which is the Turing machine based theory of computable real number functions, as it has been developed by Pour-El and Richards [11], Ko [8], Weihrauch [13]

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and others. This line of research is based on classical logic and computability is just considered as another property of classical numbers, functions and sets. In this spirit one version of the Baire Category Theorem has already been proved by Yasugi, Mori and Tsujii [14].

In the representation based approach to computable analysis, which has been developed by Weihrauch and others [13] under the name "Type-2 theory of effectivity", the computational meaning of the Baire Category Theorem can be analysed very easily. Depending on how the sequence $(A_n)_{n\in\mathbb{N}}$ is represented, i.e. how it is "given", we can compute an appropriate point x in case of the first version or compute a suitable index k in case of the second version. Roughly speaking, the second logical version requires stronger information on the sequence of sets than the first version. Unfortunately, this makes the second version of the theorem less applicable than its classical counterpart, since this strong type of information on the sequence $(A_n)_{n\in\mathbb{N}}$ is rarely available.

We close this introduction with a short survey on the organisation of the paper. In the next section we briefly summarize some basic definitions from computable analysis which will be used to formulate and prove our results. In Section 3 we discuss the first version of the computable Baire Category Theorem followed by an example of its application in Section 4, where we construct computable but nowhere differentiable functions. Finally, in Section 5 we discuss the second version of the theorem.

Further applications of the first version of the computable Baire Category Theorem can be found in [3].

2 Preliminaries from Computable Analysis

In this section we briefly summarize some notions from computable analysis. For details the interested reader is refered to [13]. The basic idea of the representation based approach to computable analysis is to represent infinite objects like real numbers, functions or sets, by infinite strings over some alphabet Σ (which should at least contain the symbols 0 and 1). Thus, a *representation* of a set X is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$ and in this situation we will call (X, δ) a *represented space*. Here the inclusion symbol is used to indicate that the mapping might be partial. If we have two represented spaces (X, δ) and (Y, δ') and a function $f :\subseteq X \to Y$, then f is called (δ, δ')*–computable*, if there exits some computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$. Of course, we have to define computability of sequence functions $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ to make this definition complete, but this can be done via Turing machines: F is computable if there exists some Turing machine, which computes infinitely long and transforms each sequence p, written on the input tape, into the corresponding sequence $F(p)$, written on the one-way output tape. Later on, we will also need computable multi-valued operations $f :\subseteq X \rightrightarrows Y$, which are defined analogously to computable functions by substituting $\delta' F(p) \in f\delta(p)$ for the equation above. If the represented spaces are fixed or clear from the context, then we will simply call a function or operation f *computable*. A *computable* *sequence* is a computable function $f : \mathbb{N} \to X$, where we assume that $\mathbb{N}$ is represented by $\delta_{\mathbb{N}}(1^n0^\omega) := n$ and a point $x \in X$ is called *computable*, if there is a constant computable function with value x.

Given two represented spaces (X, δ) and (Y, δ'), there is a canonical representation $[\delta, \delta']$ of $X \times Y$ and a representation $[\delta \to \delta']$ of certain functions $f : X \to Y$. If δ, δ' are *admissible* representations of T_0–spaces with countable bases (cf. [13]), then $[\delta \to \delta']$ is actually a representation of the set $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$. If $Y = \mathbb{R}$, then we write for short $\mathcal{C}(X) := \mathcal{C}(X, \mathbb{R})$. The function space representation can be characterized by the fact that it admits evaluation and type conversion. Evaluation means that $\mathcal{C}(X, Y) \times X \to Y, (f, x) \mapsto f(x)$ is $([[\delta \to \delta'], \delta], \delta')$–computable. Type conversion means that for any represented space (Z, δ'') a function $f : Z \to \mathcal{C}(X, Y)$ is $(\delta'', [\delta \to \delta'])$–computable, if and only if the associated function $\hat{f} : Z \times X \to Y$, defined by $\hat{f}(z, x) := f(z)(x)$, is $([\delta'', \delta], \delta')$–computable. Moreover, the $[\delta \to \delta']$–computable points are just the (δ, δ')–computable functions. Given a represented space (X, δ), we will also use the representation $\delta^{\mathbb{N}} := [\delta_{\mathbb{N}} \to \delta]$ of the set of sequences $X^{\mathbb{N}}$. Finally, we will call a subset $A \subseteq X$ *δ–r.e.*, if there exists some Turing machine that recognizes A in the following sense: whenever an input $p \in \Sigma^\omega$ with $\delta(p) \in A$ is given to the machine, the machine stops after finitely many steps, for all other $p \in \mathrm{dom}(\delta)$ it computes forever.

Many interesting representations can be derived from computable metric spaces and we will also use them to formulate the computable versions of the Baire Category Theorem.

Definition 1 (Computable metric space). A tuple (X, d, α) is called *computable metric space*, if

1. $d : X \times X \to \mathbb{R}$ is a metric on X,
2. $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X,
3. $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

Here, we tacitly assume that the reader is familiar with the notion of a computable sequence of reals, but we will come back to that point below. Obviously, a computable metric space is especially separable. Given a computable metric space (X, d, α), its *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ can be defined by

$$\delta_X(01^{n_0}01^{n_1}01^{n_2}...) := \lim_{i \to \infty} \alpha(n_i)$$

for all n_i such that $d(\alpha(n_i), \alpha(n_j)) \leq 2^{-i}$ for all $j > i$ (and undefined for all other input sequences). In the following we tacitly assume that computable metric spaces are represented by their Cauchy representation. If X is a computable metric space, then it is easy to see that $d : X \times X \to \mathbb{R}$ becomes computable. An important computable metric space is $(\mathbb{R}, d, \alpha_{\mathbb{Q}})$ with the Euclidean metric $d(x, y) := |x - y|$ and some standard numbering of the rational numbers, as $\alpha_{\mathbb{Q}}\langle i, j, k\rangle := (i - j)/(k + 1)$. Here, $\langle i, j\rangle := 1/2(i + j)(i + j + 1) + j$ denotes *Cantor pairs* and this definition is extended inductively to finite tuples. For short we will occasionally write $\overline{k} := \alpha_{\mathbb{Q}}(k)$. In the following we assume that $\mathbb{R}$ is endowed with the Cauchy representation $\delta_{\mathbb{R}}$ induced by the computable metric space given above. This representation of $\mathbb{R}$ can also be defined, if $(\mathbb{R}, d, \alpha_{\mathbb{Q}})$ just fulfills 1. and 2. of the definition above and this leads to a definition of computable real number sequences without circularity.

Other important representations cannot be deduced from computable metric spaces. Especially, we will use representations of the hyperspace of closed subsets $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed}\}$ of a metric space X, which will be defined in the following sections. For a more comprehensive discussion of hyperspace representations, see [4]. Here, we just mention that we will denote the *open balls* of (X, d) by $B(x, \varepsilon) := \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$, $\varepsilon > 0$ and correspondingly the *closed balls* by $\overline{B}(x, \varepsilon) := \{y \in X : d(x, y) \leq \varepsilon\}$. Occasionally, we denote complements of sets $A \subseteq X$ by $A^c := X \setminus A$.

3 First Computable Baire Category Theorem

For this section let (X, d, α) be some fixed complete computable metric space, and let $\mathcal{A} := \mathcal{A}(X)$ be the set of closed subsets. We can easily define a representation $\delta_{\mathcal{A}}^{>}$ of $\mathcal{A}$ by

$$\delta_{\mathcal{A}}^{>}(01^{\langle n_0, k_0 \rangle}01^{\langle n_1, k_1 \rangle}01^{\langle n_2, k_2 \rangle}...) := X \setminus \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i}).$$

We write $\mathcal{A}_>$ to indicate that we use the represented space $(\mathcal{A}, \delta_{\mathcal{A}}^{>})$. The computable points $A \in \mathcal{A}_>$ are the so-called *co-r.e. closed* subsets of X. From results in [4] it directly follows that preimages of $\{0\}$ of computable functions are computable in $\mathcal{A}_>$. We formulate the result a bit more general.

Lemma 2. *The operation $\mathcal{C}(X) \to \mathcal{A}_>, f \mapsto f^{-1}\{0\}$ is computable and admits a computable right inverse.*

Using this fact we can immediately conclude that the union operation is computable on $\mathcal{A}_>$.

Proposition 3. *The operation $\mathcal{A}_> \times \mathcal{A}_> \to \mathcal{A}_>, (A, B) \mapsto A \cup B$ is computable.*

Proof. Using evaluation and type conversion w.r.t. $[\delta_X \to \delta_{\mathbb{R}}]$, it is straightforward to show that $\mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X), (f, g) \mapsto f \cdot g$ is computable, but if $f^{-1}\{0\} = A$ and $g^{-1}\{0\} = B$, then $(f \cdot g)^{-1}\{0\} = A \cup B$. Thus the desired result follows from the previous Lemma 2. $\square$

Since computable functions have the property that they map computable points to computable points, we can deduce that the class of co-r.e. closed sets is closed under intersection.

Corollary 4. *If $A, B \subseteq X$ are co-r.e. closed, then $A \cup B$ is co-r.e. closed too.*

Moreover, it is obvious that we can compute complements of open balls in the following sense.

Proposition 5. $\left(X \setminus B(\alpha(n), \overline{k})\right)_{\langle n,k\rangle\in\mathbb{N}}$ *is a computable sequence in* $\mathcal{A}_>$.

Using these both observations, we can prove the following first version of the computable Baire Category Theorem just by transferring the classical proof.

Theorem 6 (First computable Baire Category Theorem). *There exists a computable operation* $\Delta :\subseteq \mathcal{A}_>^{\mathbb{N}} \rightrightarrows X^{\mathbb{N}}$ *with the following property: for any sequence* $(A_n)_{n\in\mathbb{N}}$ *of closed nowhere dense subsets of* X*, there exists some sequence* $(x_n)_{n\in\mathbb{N}} \in \Delta(A_n)_{n\in\mathbb{N}}$ *and all such sequences* $(x_n)_{n\in\mathbb{N}}$ *are dense in* $X \setminus \bigcup_{n=0}^{\infty} A_n$.

Proof. Let us fix some $n = \langle n_1, n_2\rangle \in \mathbb{N}$. We construct sequences $(x_{n,k})_{k\in\mathbb{N}}$ in X and $(r_{n,k})_{k\in\mathbb{N}}$ in $\mathbb{Q}$ as follows: let $x_{\langle n_1,n_2\rangle,0} := \alpha(n_1)$, $r_{\langle n_1,n_2\rangle,0} := 2^{-n_2}$. Given $r_{n,i}$ and $x_{n,i}$ we can effectively find some point $x_{n,i+1} \in \mathrm{range}(\alpha) \subseteq X$ and a rational $\varepsilon_{n,i+1}$ with $0 < \varepsilon_{n,i+1} \leq r_{n,i}$ such that

$$B(x_{n,i+1}, \varepsilon_{n,i+1}) \subseteq (X \setminus A_i) \cap B(x_{n,i}, r_{n,i}) = (A_i \cup X \setminus B(x_{n,i}, r_{n,i}))^c.$$

One the one hand, such a point and radius have to exist since A_i is nowhere dense and on the other hand, we can effectively find them, given a $\delta_{\mathcal{A}}^{>\mathbb{N}}$–name of the sequence $(A_n)_{n\in\mathbb{N}}$ and using Propositions 3 and 5. Now let $r_{n,i+1} := \varepsilon_{n,i+1}/2$. Altogether, we obtain a sequence of closed balls

$$\overline{B}(x_{n,i+1}, r_{n,i+1}) \subseteq \overline{B}(x_{n,i}, r_{n,i}) \subseteq ... \subseteq \overline{B}(x_{n,0}, r_{n,0})$$

with $r_{n,i} \leq 2^{-i}$ and thus $x_n := \lim_{i\to\infty} x_{n,i}$ exists since X is complete and the sequence $(x_{n,i})_{i\in\mathbb{N}}$ is even rapidly converging. Finally, the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in $X \setminus \bigcup_{n=0}^{\infty} A_n$, since for any pair (n_1, n_2) we obtain by definition $x_{\langle n_1,n_2\rangle} \in B(\alpha(n_1), 2^{-n_2})$. Altogether, the construction shows how a Turing machine can transform each $\delta_{\mathcal{A}}^{>\mathbb{N}}$–name of a sequence $(A_n)_{n\in\mathbb{N}}$ into a δ_X–name of a suitable sequence $(x_n)_{n\in\mathbb{N}}$. □

As a direct corollary of this uniformly computable version of the Baire Category Theorem we can conclude the following weak version.

Corollary 7. *For any computable sequence* $(A_n)_{n\in\mathbb{N}}$ *of co-r.e. closed nowhere dense subsets* $A_n \subseteq X$*, there exists some computable sequence* $(x_n)_{n\in\mathbb{N}}$ *which is dense in* $X \setminus \bigcup_{n=0}^{\infty} A_n$.

Since any computable sequence $(A_n)_{n\in\mathbb{N}}$ of co-r.e. closed nowhere dense subsets $A_n \subseteq X$ is "sequentially effectively nowhere dense" in the sense of Yasugi, Mori and Tsujii, we can conclude the previous corollary also from their effective Baire Category Theorem [14].

It is a well-known fact that the set of computable real numbers $\mathbb{R}_c$ cannot be enumerated by a computable sequence [13]. We obtain a new proof for this fact and a generalization for computable complete metric spaces without isolated points. First we prove the following simple proposition.

Proposition 8. *The operation* $X \to \mathcal{A}_>, x \mapsto \{x\}$ *is computable.*

Proof. This follows directly from the fact that $d : X \times X \to \mathbb{R}$ is computable and $\{x\} = X \setminus \bigcup\{B(\alpha(n), \overline{k}) : d(\alpha(n), x) > \overline{k}$ and $n, k \in \mathbb{N}\}$. □

If X is a metric space without isolated points, then all singleton sets $\{x\}$ are nowhere dense closed subsets. This allows to combine the previous proposition with the computable Baire Category Theorem 7.

Corollary 9. *If X is a computable complete metric space without isolated points, then for any computable sequence $(y_n)_{n\in\mathbb{N}}$ in X, there exists a computable sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_n)_{n\in\mathbb{N}}$ is dense in $X \setminus \{y_n : n \in \mathbb{N}\}$.*

Using Theorem 6 it is straightforward to derive even a uniform version of this theorem which states that we can effectively find a corresponding sequence $(x_n)_{n\in\mathbb{N}}$ for any given sequence $(y_n)_{n\in\mathbb{N}}$. Instead of formulating this uniform version, we include the following corollary which generalizes the statement that $\mathbb{R}_c$ cannot be enumerated by a computable sequence.

Corollary 10. *If X is a computable complete metric space without isolated points, then there exists no computable sequence $(y_n)_{n\in\mathbb{N}}$ such that $\{y_n : n \in \mathbb{N}\}$ is the set of computable points of X.*

4 Computable but Nowhere Differentiable Functions

In this section we want to effectivize the standard example of an application of the Baire Category Theorem. We will show that there exists a computable but nowhere differentiable function $f : [0,1] \to \mathbb{R}$. It is not to difficult to construct an example of such a function directly and actually, some typical examples of continuous nowhere differentiable functions, like *van der Waerden's function* $f : [0,1] \to \mathbb{R}$ or *Riemann's function* $g : [0,1] \to \mathbb{R}$ (cf. [9]), defined by

$$f(x) := \sum_{n=0}^{\infty} \frac{\langle 4^n x\rangle}{4^n} \text{ and } g(x) := \sum_{n=0}^{\infty} \frac{\sin(n^2\pi x)}{n^2},$$

where $\langle x\rangle := \min\{x - [x], 1 + [x] - x\}$ denotes the distance of x to the nearest integer, can easily be seen to be computable. The purpose of this section is rather to demonstrate that the computable version of the Baire Category Theorem can be applied in similar situations as the classical one.

In this section we will use the computable metric space of continuous functions $(\mathcal{C}[0,1], d_{\mathcal{C}}, \alpha_{\mathbb{Q}[x]})$, where $d_{\mathcal{C}}$ denotes the *supremum metric*, which can be defined by $d_{\mathcal{C}}(f,g) := \max_{x\in[0,1]} |f(x) - g(x)|$ and $\alpha_{\mathbb{Q}[x]}$ denotes some standard numbering of the set $\mathbb{Q}[x]$ of rational polynomials $p : [0,1] \to \mathbb{R}$. By $\delta_{\mathcal{C}}$ we denote the Cauchy representation of this space and in the following we tacitly assume that $\mathcal{C}[0,1]$ is endowed with this representation. For technical simplicity we assume that functions $f : [0,1] \to \mathbb{R}$ are actually functions $f : \mathbb{R} \to \mathbb{R}$ extended constantly, i.e. $f(x) = f(0)$ for $x \le 0$ and $f(x) = f(1)$ for $x \ge 1$. It is well-known that a function $f : [0,1] \to \mathbb{R}$ is $\delta_{\mathcal{C}}$–computable, if it is computable considered as a function $f : \mathbb{R} \to \mathbb{R}$ and we can actually replace $\delta_{\mathcal{C}}$ by the restriction of $[\delta_{\mathbb{R}} \to \delta_{\mathbb{R}}]$ to $\mathcal{C}[0,1]$ whenever it is helpful [13].

We will consider differentiability for functions $f : [0,1] \to \mathbb{R}$ only within $[0,1]$. If a function $f : [0,1] \to \mathbb{R}$ is differentiable at some point $t \in [0,1]$, then the quotient $|\frac{f(t+h)-f(t)}{h}|$ is bounded for all $h \neq 0$. Thus f belongs to the set

$$D_n := \left\{ f \in \mathcal{C}[0,1] : (\exists t \in [0,1])(\forall h \in \mathbb{R} \setminus \{0\}) \left| \frac{f(t+h)-f(t)}{h} \right| \leq n \right\}$$

for some $n \in \mathbb{N}$. Because of continuity of the functions f, it suffices if the universal quantification over h ranges over some dense subset of $\mathbb{R} \setminus \{0\}$ such as $\mathbb{Q} + \pi$ in order to obtain the same set D_n.

It is well-known, that all sets D_n are closed and nowhere dense [7]. Thus, by the classical Baire Category Theorem, the set $\mathcal{C}[0,1] \setminus \bigcup_{n=0}^{\infty} D_n$ is non-empty and there exists some continuous but nowhere differentiable function $f : [0,1] \to \mathbb{R}$. Our aim is to prove that $(D_n)_{n \in \mathbb{N}}$ is a computable sequence of co-r.e. closed nowhere dense subsets of $\mathcal{C}[0,1]$, i.e. a computable sequence in $\mathcal{A}_>(\mathcal{C}[0,1])$. Then we can apply the computable Baire Category Theorem 6 to ensure the existence of a computable but nowhere differentiable function $f : [0,1] \to \mathbb{R}$.

The crucial point is to get rid of the existential quantification of t over $[0,1]$ since arbitrary unions of co-r.e. closed sets need not to be (co-r.e.) closed again. The main tool will be the following Proposition which roughly speaking states that co-r.e. closed subsets are closed under parametrized countable and computable intersection and compact computable union.

Proposition 11. *Let (X, δ) be some represented space and let (Y, d, α) be some computable metric space.*

1. *If the function $A : X \times \mathbb{N} \to \mathcal{A}_>(Y)$ is computable, then the countable intersection $\cap A : X \to \mathcal{A}_>(Y), x \mapsto \bigcap_{n=0}^{\infty} A(x,n)$ is computable too.*
2. *If the function $U : X \times \mathbb{R} \to \mathcal{A}_>(Y)$ is computable, then the compact union $\cup U : X \to \mathcal{A}_>(Y), x \mapsto \bigcup_{t \in [0,1]} U(x,t)$ is computable too.*

Proof. 1. Let $A : X \times \mathbb{N} \to \mathcal{A}_>(Y)$ be computable. If for some fixed $x \in X$ we have $A(x,n) = Y \setminus \bigcup_{k=0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}})$ with $i_{nk}, j_{nk} \in \mathbb{N}$ for all $n, k \in \mathbb{N}$, then

$$\bigcap_{n=0}^{\infty} A(x,n) = \bigcap_{n=0}^{\infty} \left(Y \setminus \bigcup_{k=0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}}) \right) = Y \setminus \left(\bigcup_{\langle n,k \rangle = 0}^{\infty} B(\alpha(i_{nk}), \overline{j_{nk}}) \right).$$

Thus, it is straightforward to show that $\cap A : X \to \mathcal{A}_>(Y)$ is computable too.

2. Now let $U : X \times \mathbb{R} \to \mathcal{A}_>(Y)$ be computable. Let $\delta_{[0,1]} :\subseteq \Sigma^\omega \to [0,1]$ be the *signed digit representation* of the unit interval, where $\Sigma = \{0, 1, -1\}$ and $\delta_{[0,1]}$

is defined in all possible cases by

$$\delta_{[0,1]}(p) := \sum_{i=0}^{\infty} p(i)2^{-i}.$$

It is known that $\mathrm{dom}(\delta_{[0,1]})$ is compact and $\delta_{[0,1]}$ is computably equivalent to the Cauchy representation $\delta_{\mathbb{R}}$, restricted to $[0,1]$ (cf. [13]). Thus, U restricted to $X\times[0,1]$ is $([\delta,\delta_{[0,1]}],\delta_{\mathcal{A}}^{>})$–computable. Then there exists some Turing machine M which computes a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ which is a $([\delta,\delta_{[0,1]}],\delta_{\mathcal{A}}^{>})$–realization of $U : X \times \mathbb{R} \to \mathcal{A}_>(Y)$. Thus, for each given input sequence $\langle p,q\rangle \in \Sigma^\omega$ with $x := \delta(p)$ and $t := \delta_{[0,1]}(q)$ the machine M produces some output sequence $01^{\langle n_{q0},k_{q0}\rangle}01^{\langle n_{q1},k_{q1}\rangle}01^{\langle n_{q2},k_{q2}\rangle}...$ such that

$$U(x,t) = Y \setminus \bigcup_{i=0}^{\infty} B(\alpha(n_{qi}),\overline{k_{qi}}).$$

Since we will only consider a fixed p, we do not mention the corresponding dependence in the indices of the values n_{qi}, k_{qi}. It is easy to prove that the set $W := \{w \in \Sigma^* : (\exists q \in \mathrm{dom}(\delta_{[0,1]}))\ w \text{ is a prefix of } q\}$ is recursive.

We will sketch the construction of a machine M' which computes the operation $\cup U : X \to \mathcal{A}_>(Y)$. On input p the machine M' works in parallel phases $\langle i,j,k\rangle = 0,1,2,...$ and produces an output r. In phase $\langle i,j,k\rangle$ it simulates M on input $\langle p, w0^\omega\rangle$ for all words $w \in \Sigma^k \cap W$ and exactly k steps. Let $01^{\langle n_{w0},k_{w0}\rangle}01^{\langle n_{w1},k_{w1}\rangle}...01^{\langle n_{wl_w},k_{wl_w}\rangle}0$ be the corresponding output of M (more precisely: the longest prefix of the output which ends with 0). Then the machine M' checks whether for all $w \in \Sigma^k \cap W$ there is some $\iota_w = 0,...,l_w$ such that $d(\alpha(i),\alpha(n_{w\iota_w})) + \overline{j} < \overline{k_{w\iota_w}}$ holds, which especially implies

$$B(\alpha(i),\overline{j}) \subseteq \bigcap_{w\in\Sigma^k\cap W} B(\alpha(n_{w\iota_w}),\overline{k_{w\iota_w}}) \subseteq \bigcap_{t\in[0,1]} Y\setminus U(x,t) = Y \setminus \bigcup U(x).$$

The verification is possible since (X,d,α) is a computable metric space. As soon as corresponding values ι_w are found for all $w \in \Sigma^k\cap W$, phase $\langle i,j,k\rangle$ is finished with extending the output by $01^{\langle i,j\rangle}$. Otherwise it might happen that the phase never stops, but other phases may run in parallel.

We claim that this machine M' actually computes $\cup U$. On the one hand, it is clear that $B(\alpha(i),\overline{j}) \subseteq Y \setminus \cup U(x)$ whenever $01^{\langle i,j\rangle}$ is written on the output tape by M'. Thus, if M' actually produces an infinite output r, then we obtain immediately $\delta_{\mathcal{A}}^{>}(r) \subseteq \cup U(\delta(p))$. On the other hand, let $y \in Y \setminus \cup U(\delta(p))$. Then for any $q \in \mathrm{dom}(\delta_{[0,1]})$ the machine M produces some output sequence $01^{\langle n_{q0},k_{q0}\rangle}01^{\langle n_{q1},k_{q1}\rangle}01^{\langle n_{q2},k_{q2}\rangle}...$ and there has to be some l_q such that $y \in B(\alpha(n_{ql_q}),\overline{k_{ql_q}})$ and a finite number k of steps such that M produces $01^{\langle n_{ql_q},k_{ql_q}\rangle}0$ on the output tape. Since $\mathrm{dom}(\delta_{[0,1]})$ is compact, there is even a common such k for all $q \in \mathrm{dom}(\delta_{[0,1]})$. Let $w' := w0^\omega$ for all $w \in \Sigma^*$. Then there exist $i,j \in \mathbb{N}$ such that

$$y \in B(\alpha(i),\overline{j}) \subseteq \bigcap_{w\in\Sigma^k\cap W} B(\alpha(n_{w'l_{w'}}),\overline{k_{w'l_{w'}}})$$

and $d(\alpha(i), \alpha(n_{w'l_{w'}})) + \overline{j} < \overline{k_{w'l_{w'}}}$. Thus M' will produce $01^{\langle i,j\rangle}$ on the output tape in phase $\langle i,j,k\rangle$. Altogether, this proves $\delta_{\mathcal{A}}^{>}(r) = \cup U(\delta(p))$ and thus the operation $\cup U : X \to \mathcal{A}_>(Y)$ is computable. □

Now using this proposition, we can directly prove the desired result.

Theorem 12. *There exists a computable sequence $(f_n)_{n\in\mathbb{N}}$ of computable but nowhere differentiable functions $f_n : [0,1] \to \mathbb{R}$ such that $\{f_n : n \in \mathbb{N}\}$ is dense in $\mathcal{C}[0,1]$.*

Proof. If we can prove that $(D_n)_{n\in\mathbb{N}}$ is a computable sequence of co-r.e. nowhere dense closed sets, then Corollary 7 implies the existence of a computable sequence of computable functions f_n in $\mathcal{C}[0,1] \setminus \bigcup_{n=0}^{\infty} D_n$. Since all somewhere differentiable functions are included in some D_n, it follows that all f_n are nowhere differentiable. Since it is well-known that all D_n are nowhere dense, it suffices to prove the computability property. We recall that it suffice to consider values $h \in \mathbb{Q} + \pi$ in the definition of D_n because of continuity of the functions f. We define a function $F : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \times \mathcal{C}[0,1] \to \mathbb{R}$ by

$$F(n,t,k,f) := \max\left\{\left|\frac{f(t+\overline{k}+\pi) - f(t)}{\overline{k}+\pi}\right| - n, 0\right\}.$$

Then using the evaluation property of $[\delta_{\mathbb{R}} \to \delta_{\mathbb{R}}]$, one can prove that F is computable. Using type conversion w.r.t. $[\delta_{\mathcal{C}} \to \delta_{\mathbb{R}}]$ one obtains computability of $\hat{F} : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \to \mathcal{C}(\mathcal{C}[0,1])$, defined by $\hat{F}(n,t,k)(f) := F(n,t,k,f)$. Using Lemma 2 we can conclude that the mapping $A : \mathbb{N} \times \mathbb{R} \times \mathbb{N} \to \mathcal{A}_>(\mathcal{C}[0,1])$ with $A(n,t,k) := (\hat{F}(n,t,k))^{-1}\{0\}$ is computable. Thus by the previous proposition $\cap A : \mathbb{N} \times \mathbb{R} \to \mathcal{A}_>(\mathcal{C}[0,1])$ is also computable and thus $\cup \cap A : \mathbb{N} \to \mathcal{A}_>(\mathcal{C}[0,1])$ too. Now we obtain

$$\cup \cap A(n) = \bigcup_{t\in[0,1]} \bigcap_{k=0}^{\infty} \left\{f \in \mathcal{C}[0,1] : \left|\frac{f(t+\overline{k}+\pi) - f(t)}{\overline{k}+\pi}\right| \le n\right\} = D_n.$$

Thus, $(D_n)_{n\in\mathbb{N}}$ is a computable sequence of co-r.e. closed subsets of $\mathcal{C}[0,1]$. □

5 Second Computable Baire Category Theorem

While the first version of the computable Baire Category Theorem has been proved by a direct adaptation of the classical proof, the second version will even be a consequence of the classical version. Whenever a classical theorem for complete computable metric spaces X, Y has the form

$$(\forall x)(\exists y)R(x,y)$$

with a predicate $R \subseteq X \times Y$ which can be proven to be r.e. open, then the theorem admits a computable multi-valued realization $F : X \rightrightarrows Y$ such that $R(x,y)$ holds for all $y \in F(x)$ (cf. the Uniformization Theorem 3.2.40 in [2]).

Actually, a computable version of the second formulation of the Baire Category Theorem, given in the Introduction, can be derived as such a direct corollary of the classical version.

Given a co-r.e. set $A \subseteq X$, the closure of its complement $\overline{A^c}$ needs not to be co-r.e. again (cf. Proposition 5.4 in [1]). Thus, the operation $\mathcal{A}_> \to \mathcal{A}_>, A \mapsto \overline{A^c}$ cannot be computable (and actually it is not even continuous in the corresponding way). In order to overcome this deficiency, we can simply include the information on $\overline{A^c}$ into a representing sequence of A. This is a usual trick in topology and computable analysis to make functions continuous or computable, respectively. So, if δ is an arbitrary representation of $\mathcal{A}$, then the representation δ^+ of $\mathcal{A}$, defined by

$$\delta^+\langle p, q\rangle := A :\iff \delta(p) = A \text{ and } \delta_{\mathcal{A}}^{>}(q) = \overline{A^c},$$

has automatically the property that $\mathcal{A} \to \mathcal{A}, A \mapsto \overline{A^c}$ becomes $(\delta^+, \delta_{\mathcal{A}}^{>})$–computable. Here $\langle\ \rangle : \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ denotes some appropriate computable pairing function [13]. We can especially apply this procedure to $\delta := \delta_{\mathcal{A}}^{>}$. The corresponding $\delta_{\mathcal{A}}^{>+}$–computable sets $A \subseteq X$ are called *bi-co-r.e.* closed sets. In this case we write $\mathcal{A}_{>+}$ to denote the represented space $(\mathcal{A}, \delta_{\mathcal{A}}^{>+})$. Now we can directly conclude that the property "somewhere dense" is r.e.

Proposition 13. *The set $\{A \in \mathcal{A} : A$ is somewhere dense$\}$ is r.e. in $\mathcal{A}_{>+}$.*

The proof follows directly from the fact that a closed set $A \subseteq X$ is somewhere dense, if and only if there exist $n, k \in \mathbb{N}$ such that $B(\alpha(n), \overline{k}) \subseteq A^\circ = \overline{A^c}^{\,c}$. We can now directly conclude the second computable version of the Baire Category Theorem as a consequence of the classical version (and thus especially as a consequence of the first computable Baire Category Theorem 6).

Theorem 14 (Second computable Baire Category Theorem). *There exists a computable operation $\Sigma :\subseteq \mathcal{A}_{>+}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ with the following property: for any sequence $(A_n)_{n\in\mathbb{N}}$ of closed subsets of X with $X = \bigcup_{n=0}^{\infty} A_n$, there exists some $\langle i, j, k\rangle \in \Sigma(A_n)_{n\in\mathbb{N}}$ and for all such $\langle i, j, k\rangle$ we obtain $B(\alpha(i), \overline{j}) \subseteq A_k$.*

Of course, if we replace $\mathcal{A}_{>+}$ by $(\mathcal{A}, \delta^+)$ with any other underlying representation δ instead of $\delta_{\mathcal{A}}^{>}$, then the theorem would also hold true. We mention that the corresponding constructive version of the theorem (Theorem 2.5 in [6]), if directly translated into a computable version, leads to a weaker statement than Theorem 14: if the sequence $(A_n)_{n\in\mathbb{N}}$ would be effectively given by the sequences of distance functions of A and A^c, this would constitute a stronger input information than it is the case if it is given by $\delta_{\mathcal{A}}^{>+}$. Now we can formulate a weak version of the second Baire Category Theorem.

Corollary 15. *For any computable sequence $(A_n)_{n\in\mathbb{N}}$ of bi-co-r.e. closed subsets $A_n \subseteq X$ with $X = \bigcup_{j=0}^{\infty} A_{\langle i,j\rangle}$ for all $i \in \mathbb{N}$, there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $A_{\langle i, f(i)\rangle}$ is somewhere dense for all $i \in \mathbb{N}$.*

By applying some techniques from recursion theory [12, 10], we can prove that the previous theorem and its corollary do not hold true with $\mathcal{A}_>$ instead of $\mathcal{A}_{>+}$. For this result we use as metric space the Euclidean space $X = \mathbb{R}$.

Theorem 16. *There exists a computable sequence $(A_n)_{n\in\mathbb{N}}$ of co-r.e. closed subsets $A_n \subseteq [0,1]$ with $[0,1] = \bigcup_{j=0}^\infty A_{\langle i,j\rangle}$ for all $i \in \mathbb{N}$ such that for every computable $f : \mathbb{N} \to \mathbb{N}$ there is some $i \in \mathbb{N}$ such that $A_{\langle i,f(i)\rangle}$ is nowhere dense.*

Proof. We use some total Gödel numbering $\varphi : \mathbb{N} \to P$ of the set of partial recursive functions $P := \{f :\subseteq \mathbb{N} \to \mathbb{N} : f \text{ computable}\}$ to define sets

$$A'_{\langle i,j\rangle} := \overline{\bigcup_{k=0}^{\min \varphi_i^{-1}\{j\}} \left\{\frac{m}{2^k} : m = 0, ..., 2^k\right\}}.$$

For this definition we assume $\min \emptyset = \infty$. Whenever $i \in \mathbb{N}$ is the index of some total recursive function $\varphi_i : \mathbb{N} \to \mathbb{N}$ such that $\text{range}(\varphi_i) \neq \mathbb{N}$, then we obtain $\bigcup_{j=0}^\infty A'_{\langle i,j\rangle} = [0,1]$ and $A'_{\langle i,j\rangle}$ is somewhere dense, if and only if $j \notin \text{range}(\varphi_i)$. Using the smn-Theorem one can inductively prove that there is a total recursive function $r : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{r\langle i,j\rangle}$ is total if φ_i is and

$$\text{range}(\varphi_{r\langle i,\langle k,\langle n_0,...,n_k\rangle\rangle\rangle}) = \text{range}(\varphi_i) \cup \{n_0, ..., n_k\}.$$

Let i_0 be the index of some total recursive function which enumerates some simple set $S := \text{range}(\varphi_{i_0})$ and define $A_{\langle i,j\rangle} := A'_{\langle r\langle i_0,i\rangle,j\rangle}$. Then $(A_n)_{n\in\mathbb{N}}$ is a computable sequence of co-r.e. closed subsets $A_n \subseteq [0,1]$. Let us assume that there exists a total recursive function $f : \mathbb{N} \to \mathbb{N}$ with the property that $A_{\langle i,f(i)\rangle}$ is somewhere dense for all $i \in \mathbb{N}$. Let $j_0 \in \mathbb{N} \setminus S$ and define a function $g : \mathbb{N} \to \mathbb{N}$ inductively by $g(0) := j_0$ and $g(n+1) := f(r\langle i_0, \langle n, \langle g(0), ...g(n)\rangle\rangle\rangle)$. Then g is computable and $\text{range}(g)$ is some infinite r.e. subset of the immune set $\mathbb{N} \setminus S$. Contradiction! □

The reader might notice that the constructed sequence $(A_n)_{n\in\mathbb{N}}$ is even a computable sequence of recursive closed sets (cf. [5, 13]). Even a simpler variant of the same idea can be used to prove that in a well-defined sense there exists no continuous multi-valued operation $\Sigma :\subseteq \mathcal{A}_>^{\mathbb{N}} \rightrightarrows \mathbb{N}$ which meets the conditions of Theorem 14.

Unfortunately, the simplicity of the proof of the second computable Baire Category Theorem 14 corresponds to its uselessness. The type of information that one could hope to gain from an application of the theorem has already to be fed in by the input information. However, Theorem 16 shows that a substantial improvement of Theorem 14 seems to be impossible.

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